The Noether Symmetries of the Lagrangians of Spacetimes

by

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A Thesis

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Abstract

In this thesis Noether symmetries are used for the classification of plane symmetric, cylindrically symmetric and spherically symmetric static spacetimes. We consider general metrics for these spacetimes and use their general arc length minimizing Lagrangian densities for the classification purpose. The coefficients of the metric in case of plane symmetric static spacetime are general functions of x while the coefficients of cylindrically symmetric and spherically symmetric static spacetimes are general functions of the radial coordinate r. The famous Noether symmetry equation is used for the arc length minimizing Lagrangian densities of these spacetimes. Noether symmetries and particular arc length minimizing Lagrangian densities of plane symmetric, cylindrically symmetric and spherically symmetric static spacetimes are obtained. Once we get the particular Lagrangian densities, we can obtain the corresponding particular spacetimes easily. This thesis not only provides classification of the spacetimes but we can also obtain first integrals corresponding to each Noether symmetry. These first integrals can be used to define conservation laws in each spacetime.

By using general arc length minimizing Lagrangian for plane symmetric, cylindrically symmetric and spherically symmetric static spacetimes in the Noether symmetry equation a system of 19 partial differential equations is obtained in each case. The solution of the system in each case provides us three important things; the classification of the spacetimes, the Noether symmetries and the corresponding first integrals which can be used for the conservation laws relative to each spacetime.

Energy and momentum, the definitions of which are the focus of many investigations in general relativity, are important quantities in physics. Since there is no invariant definitions of energy and momentum in general relativity to define these quantities we use the

approximate Noether symmetries of the general geodesic Lagrangian density of the general time conformal plane symmetric spacetime. We use approximate Noether symmetry condition for this purpose to calculate the approximate Noether symmetries of the action of the Lagrangian density of time conformal plane symmetric spacetime. From this approach, those spacetimes are obtained the actions of which admit the first order approximation. The corresponding spacetimes are the approximate gravitational wave spacetimes which give us information and insights for the exact gravitational wave spacetimes. Some of the Noether symmetries obtained here carry approximate parts. These approximate Noether symmetries can further be used to find the corresponding first integrals which describe the conservation laws in the respective spacetimes.

Some of the vacuum solutions of Einstein field equations for plane symmetric, cylindrically symmetric and spherically symmetric static spacetimes have also been explored.

Dedicated to

My Mother

and

My Father Muhammad Gul (late)

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List of Publications from This Thesis

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Contents

Chapter 1

Introduction

In 1867, a Norwegian mathematician, Sophus Lie, introduced a powerful technique for the solutions of differential equations [8,10,11,36,42,59]. The beauty of his technique is that it is applicable to all types of differential equations i.e. homogenous, non-homogeneous, linear, non-linear, ordinary and partial differential equations of order n. Later on, he used this technique for the linearization of non-linear differential equations, group classification of differential equations, and for finding the invariants corresponding to differential equations.

1.1 Sophus Lie

Sophus Lie was born on 17 December 1842 at Nordfjordeid Norway [81]. He joined a school in the town of Moss, which is a port in the south east of Norway. In 1857, he joined Nessen's Private Latin school in Christiania, where he decided to join army, but due to his weak eye-sight he gave up this idea and got admission in the university of Christiania. In the university, he studied a broad science course. There he attended the lectures of Ludwig Sylow on the work of Galois on algebraic equations and the lectures given by Carl Bjerknes. Sophus Lie graduated in 1865 from the same university, without showing any great ability for Mathematics or any liking for this subject. But afterward he made a mathematical career. The first brilliant idea that came to his mind in 1867 was to develop new geometries by considering curves rather than points only. This idea was further developed after he studied papers on geometry by Pluker and Poncelet. Lie composed a short scientific paper on this new idea in 1869, and published it at his own expense. He wrote a more definitive work but the Academy of Science in Christiania did not publish it. Later he submitted it to the Crelle Journal where it was accepted. He sent letters to two Prussian mathematicians, Reye and Clebsch, about his work. For the paper in Crelle Journal, Lie won a grant to travel and meet the leading mathematicians. By the end of the year 1869, Lie went to Prussia, then Gottingen, and after that to Berlin. In Berlin, he met Kronecker, Kummer and Weierstrass. He was not attracted by the style of Weierstrass who was the leading mathematician of Berlin, but his ideas matched with that of Kummer. Lie presented his research work in Kummer's workshop and was able to revise a few lapses that Kummer had made in his work on line congruences of degree three. The most important to Lie was, that in Berlin he met Felix Klein. It was not difficult to see that these two (Lie and Klein) would in fact have the same mathematical background, since Klein had been a student of Plucker, and Lie, despite the fact that he never met Plucker, always said that he felt like a Plucker-student. Regardless of the basic connection through Plucker's line geometry, Lie and Klein were somewhat distinctive characters as humans and mathematicians. The algebraist Klein was fascinated by the peculiarities of charming problems; the analyst Lie, parting from special cases, sought to understand a problem in its appropriate generalisation [81].

In Berlin Lie gained confidence in his research. He received high appreciation from Kummer, and he also got answers from Reye and Clebsch to his prior letters which significantly encouraged him. In the spring of 1870 Lie and Klein met in Paris. There they met Darboux, Chasles and Camille Jordan who were the leading mathematicians at the time, especially Jordan was an expert in Galois theory.

Lie started to examine these new plans on groups and geometry with Klein and they wrote many papers in this area of research. It was the winter of 1873-74 that Lie developed effectively what is called Lie Group Transformation. Later on, Killing independently investigated the algebra corresponding to these groups and Cartan completed the classification of semi-simple Lie algebras in 1900.

In short, Sophus Lie was a great mathematician and known as the founder of Lie group analysis. This analysis unified three branches of Mathematics namely algebra, analysis and geometry. Lie gained the idea of transformation groups from Galois Theory, which is the group theoretic approach to the solution of algebraic equations. Galois Theory associates permutation groups to the solution of algebraic equations. Lie applied this idea to the solution of differential equations and he claimed that there will be groups of transformations associated to the solution of differential equations.

1.2 Symmetries

The meaning of symmetry, in a vague sense is the "harmonious, beautiful proportion and balance" of a body. The reflection symmetry is the simplest one which in the language of Mathematics is called line symmetry or mirror symmetry. As an example, it is easy to observe that apparently one half of a body is exactly equal and of the same shape as the other half of the body. In mathematical language beautiful proportion and balance is "patterned self similarity" that can be written in some formal mathematical expression. Formally, symmetry in mathematics is a transformation that leaves the object unchanged. For example, symmetries of functions, differential equations, integral equations etc. are transformations of the variables which leave the functions, differential equations, integral equations, etc. unchanged. In the following table a simple examples for each case are given.

Table 1.1: Examples of symmetries

S.N	Types	Examples	Symmetry Transformation
1	Algebraic expression	$x^2 + y^2$	$(x, y) \to (x\cos\epsilon - y\sin\epsilon, x\sin\epsilon + y\cos\epsilon)$
2	Differential equation	$\frac{dy}{dx} = x - y$	$(x, y) \to (x + \epsilon, y + \epsilon)$
3	Integral equation	$I = \int \sqrt{1 + (u_x)^2}dx$	$(x, u) \to (x\cos\epsilon - u\sin\epsilon, x\sin\epsilon + u\cos\epsilon)$

All the transformations given in the above table depend upon a parameter $\epsilon \in \mathbb{R}$, and are called Lie groups of point transformations.

In the last decade of the nineteenth century Sophus Lie presented the idea of continuous groups. He proved that the order of differential equations can be reduced by one if it is invariant under a one parameter Lie group of point transformations. Lie developed these continuous Lie groups, which are now used as a mathematical tool for the solution of differential equations and symmetry based mathematical sciences. Lie point symmetry of a system is a symmetry transformation that maps solutions of the system on to solutions of the system, i.e. it maps the set of solutions of the system to itself. Translations, rotations and scaling are examples of the Lie point symmetry transformation. For an object O [42], the set S which contains all invertible transformations T leaving O invariant is called symmetry group of the object O that is

$$T : O \to O,$$

such that the set S contains identity, I, the inverse transformations T^{-1}, for all $T \in S$ and the composition $T_1T_2 \in S$ of the transformations T_1, $T_2 \in S$.

Lie used Galois's idea of groups and developed his group theoretic technique for the solution of differential equations. Galois groups are finite but Lie symmetry groups contain infinitely many transformations that depend upon continuous parameters. The main idea of a Lie group of transformations is that it employs infinitesimal transformations which form a vector space closed under a Lie algebra. Lie groups are smooth and twice differentiable manifolds which can be studied using differential calculus in contrast to the case of more general topological groups. One of the key ideas in the theory of Lie groups is to replace the global object, that is the group, with its local or linearized version which Lie himself called its "infinitesimal group" and which has become known as its Lie algebra. To understand Lie groups and Lie algebras, it is necessary to understand the concepts of manifold, tangent space, tangent bundle, etc., for which a brief introduction to all these concepts are given below.

1.3 Manifold, Tangent Space and Tangent Bundle

Manifold: The manifold is one of the most basic concept in mathematical physics [59]. It bears the idea of the space which may be curved and has some complicated topology, but it looks like Euclidean space $\mathbb{R}^n$ locally (it does not mean that both have the same metric). To apply calculus, a manifold is divided into coordinate charts which form an atlas. Union of these coordinate charts form the manifold back.

Definition: An $n-$dimensional manifold is a non empty set S, with countable subsets $s_i \subset S$, called the coordinate charts, and one to one functions $f_i : s_i \to v_i$ onto connected subsets $v_i \subset \mathbb{R}^n$, called local coordinate mapping which satisfy the following conditions.
(1): The coordinate charts cover S; that is

$$\bigcup_i s_i = S.$$

(2): On the intersection of coordinate charts, $s_i \bigcap s_j$, the composition function

$$f_i o f_j^{-1} : f_i(s_i \bigcap s_j) \to f_j(s_i \bigcap s_j),$$

is smooth (infinitely differentiable).
(3): For distinct points $p \in s_i$, $q \in s_j$ in S, there exist open subsets $w \subset v_i$, $y \subset v_j$, with the property $f_i(p) \in w$, $f_j(q) \in y$, satisfying

$$f_i^{-1}(w) \bigcap f_j^{-1}(y) = \phi.$$

Example 1: The simplest m-dimensional manifold is the Euclidean space $\mathbb{R}^m$. It is covered by a single coordinate chart $U = \mathbb{R}^m$, with local coordinate identity map given by

$$I : U = \mathbb{R}^m \to \mathbb{R}^m.$$

More generally, any open subset U of $\mathbb{R}^m$ is an m-dimensional manifold with a coordinate chart given by U itself, and with local coordinate identity map

$$I : U \to U \subset \mathbb{R}^m.$$

Sub-manifold: Given a smooth manifold S, a sub-manifold $N \subset S$ should be a subset of S satisfying all the conditions of a manifold. The unit circle $S^1 := x^2 + y^2 = 1$ and the unit sphere $S^2 = x^2 + y^2 + z^2 = 1$ are examples of 1-dimensional and 2-dimensional sub-manifolds of $\mathbb{R}^m, m \geq 2$, respectively. More specifically we have the following definition of sub-manifold.

Definition: Let S be a smooth manifold. A sub-manifold is subset $N \subset S$, along with a smooth one to one map,

$$\phi : \tilde{N} \to N \subset S,$$

satisfying the maximal rank condition every where, and $\tilde{N}$ is another manifold such that $N = \phi(\tilde{N})$. In particular, the dimension of N is the same as of $\tilde{N}$, and does not exceed the dimension of S. Here the maximal rank of the map mean that there is no singularity on the manifold N.

Tangent space to a manifold: The collection of all tangent vectors at point $p \in S$ to all possible curves passing through this point is called the tangent space to S at point p denoted by $TS|_p$. If S is an n dimensional manifold then $TS|_p$ is also an n dimensional space generated by the basis vectors

$$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}.$$

Tangent bundle: The union of all possible tangent spaces over the manifold S is called the tangent bundle of S that is

$$TS := \bigcup_{p \in S} TS|_p.$$

Vector field: A vector field $\mathbf{X}$ on S is a function that assigns a tangent vector $\mathbf{X}|_\mathbf{p}$ to each point $p \in S$, where $\mathbf{X}|_\mathbf{p}$ varies smoothly form point to point on the manifold S. If we have the local coordinates $\mathbf{x} := (x^1, x^2, ..., x^n)$, then the vector field takes the form

$$\mathbf{X}|_\mathbf{p} = \xi^1(\mathbf{x})\frac{\partial}{\partial x^1} + \xi^2(\mathbf{x})\frac{\partial}{\partial x^2} + ... + \xi^n(\mathbf{x})\frac{\partial}{\partial x^n},$$

where all $\xi^i(x)$ are smooth functions of $\mathbf{x}$.

Lie Brackets: For the vector fields $\mathbf{X_1}$ and $\mathbf{X_2}$ on the manifold S, the Lie bracket is defined as

$$[\mathbf{X_1}, \mathbf{X_2}].\phi = \mathbf{X_1}(\mathbf{X_2}.\phi) - \mathbf{X_2}(\mathbf{X_1}.\phi),$$

for all smooth function

$$\phi : S \to \mathbb{R}.$$

The Lie bracket satisfied the following properties,

(1): **Bilinearity**: For vector fields $\mathbf{X_1}$, $\mathbf{X_2}$, $\mathbf{X_3}$, $\mathbf{X_4}$ on any manifold and constants c_1, c_2, c_3, c_4 the bilinearity condition is

$$[c_1\mathbf{X_1} + c_2\mathbf{X_2}, c_3\mathbf{X_3} + c_4\mathbf{X_4}] = c_1c_3[\mathbf{X_1}, \mathbf{X_3}] + c_1c_4[\mathbf{X_1}, \mathbf{X_4}] + c_2c_3[\mathbf{X_2}, \mathbf{X_3}] + c_2c_4[\mathbf{X_2}, \mathbf{X_4}].$$

(2): **Skew symmetry**: For vector fields $\mathbf{X_1}$ and $\mathbf{X_2}$ on a manifold the condition

$$[\mathbf{X_1}, \mathbf{X_2}] = -[\mathbf{X_2}, \mathbf{X_1}],$$

holds and is called skew symmetry.

(3): **Jacobi identity**: If $\mathbf{X_1}$, $\mathbf{X_2}$ and $\mathbf{X_3}$ are vector fields on a manifold then they satisfy the condition

$$[\mathbf{X_1}, [\mathbf{X_2}, \mathbf{X_3}]] + [\mathbf{X_2}, [\mathbf{X_3}, \mathbf{X_1}]] + [\mathbf{X_3}, [\mathbf{X_1}, \mathbf{X_2}]] = 0,$$

which is called the Jacobi identity.

1.3.1 Lie Group

Lie groups are in fact manifolds, they satisfy all the conditions of the manifolds. These groups arise as an algebraic abstraction of the notion of symmetry transformations called Lie symmetry transformations; an important example is the group of rotations in the plane or three-dimensional space. Manifolds, which form the fundamental objects in the field of differential geometry, generalize the familiar concepts of curves and surfaces in three-dimensional space. In general, a manifold is a space that locally looks like Euclidean space $\mathbb{R}^m$, but the global character of which might be quite different. The conjunction of these two seemingly disparate mathematical ideas combines, and significantly extends, both the algebraic methods of group theory and the multi-variable calculus used in analytic geometry. This resulting theory, particularly the powerful infinitesimal symmetry generators techniques, can then be applied to a wide range of physical and mathematical problems.

Definition: An r-dimensional Lie group is defined as a group G which carries the structure of an r-dimensional manifold in such a way that the following composition function f and inversion function k are smooth for all elements of G

$$\begin{aligned} f: &\quad G \times G \to G, \quad f(g,h) = g.h, \quad g,h \in G, \\ k: &\quad G \to G, \quad k(g) = g^{-1}, \quad g \in G. \end{aligned}$$

Example 2: (i) $G = \mathbb{R}$ is the set all real numbers which satisfy all the conditions of Lie groups under addition, it is a simple example of a Lie group.

(ii) The group $G = GL(n, \mathbb{R})$ is the set of all $n \times n$ non singular matrices with real entries.

These matrices form a Lie group under matrix multiplication. The product of two non singular matrices is again a non singular matrix, the inverse of each matrix exists as it is non singular, the identity matrix is the identity of the group and matrix multiplication is always associative.

(iii): The set $SO(2,\mathbb{R})$ is the set of 2×2 special orthogonal matrices of the form

$$G := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad 0 \leq \theta \leq 2\pi$$

which is the rotation group in $\mathbb{R}^{\mathbf{2}}$. These matrices form Lie group.

Lie subgroup: Most often Lie group arises as subgroup of a larger group, for example the orthogonal group $SO(2,\mathbb{R})$ of 2×2 matrices with determinant equal to 1, is the subgroup of the general linear group $GL(2,\mathbb{R})$ of all invertible 2×2 matrices, similarly the orthogonal group $SO(n,\mathbb{R})$ is subgroup of of group $GL(n,\mathbb{R})$ of general linear invertible matrices. Lie sub groups are groups in their own right.

1.3.2 Lie algebra

For Lie group G, there are certain distinguished vector fields on G characterized by their invariance under the group multiplication. These invariant vector fields form a finite-dimensional vector space, called the Lie algebra of G denoted by g, which are in a precise sense the "infinitesimal generators" of G. Almost all the information in the group G is contained in its Lie algebra. This fundamental observation is the cornerstone of Lie group theory; for example, it enables us to replace complicated nonlinear conditions of invariance under a group action by relatively simple linear infinitesimal conditions. The power of this method cannot be overestimated indeed almost the entire range of applications of Lie groups to differential equations ultimately rests on this one construction.

Definition: An r-parameter Lie group G has an r-dimensional Lie algebra forming a vector space and denoted by "g" which contains all the generators of an r-dimensional Lie group, satisfying the conditions of **bilinearity, skew symmetry and Jacobi identity** defined above. This algebra is said to be **abelian** if $[\mathbf{X_i}, \mathbf{X_j}] = 0$ for all $\mathbf{X_i}, \mathbf{X_j} \in g$.

1.3.3 Lie Derivative, Isometry and Homothety

Let $\mathbf{v}$ be a vector field on a manifold S [59]. We are often interested in how certain geometric objects on S, such as functions, tensor, differential forms and other vector fields, vary under the flow $exp(\epsilon\mathbf{v})$ induced by $\mathbf{v}$. The Lie derivative of such an object will in effect tell us its infinitesimal change when acted on by the flow. (Our standard integration procedures will tell us how to reconstruct the variation under the flow from this infinitesimal version.) For instance, the behaviour of a function f under the flow induced by a vector field $\mathbf{v}$ is $\mathbf{v}(f)$, and will be the "Lie derivative" of the function f with respect to $\mathbf{v}$.

More specifically let $\mathbf{X}$ be a vector field on the manifold S and υ is another field or differential form then the Lie derivative of υ with respect to $\mathbf{X}$ at point $p \in S$ such that the following limit holds is

$$\mathcal{L}_{\mathbf{X}}(\upsilon) = \mathbf{X}(\upsilon)|_p = \lim_{\epsilon \to 0} \frac{\phi(\upsilon|_{exp(\epsilon\mathbf{X})p}) - \upsilon|_p}{\epsilon}.$$

For two vector fields $\mathbf{X_1}$ and $\mathbf{X_2}$ the Lie derivative of $\mathbf{X_2}$ with respect to $\mathbf{X_1}$ is in fact the Lie bracket

$$[\mathbf{X_1}, \mathbf{X_2}] = \mathbf{X_1}(\mathbf{X_2}) - \mathbf{X_2}(\mathbf{X_1}).$$

Isometry: [56] The field $\mathbf{X}$ on the manifold S is an isometry if the Lie derivative of the metric tensor $g_{\mu\nu}$ of S with respect to the field $\mathbf{X}$ vanishes, that is

$$\mathcal{L}_{\mathbf{X}}(g_{\mu\nu}) := g_{\mu\nu,\lambda}\mathbf{X}^{\lambda} + g_{\mu\lambda}\mathbf{X}^{\lambda}{}_{\nu} + g_{\nu\lambda}\mathbf{X}^{\lambda}{}_{\mu} = 0, \quad (\mu, \nu, \lambda = 0, 1, 2, 3),$$

where $g_{\mu\nu}$ are the coefficient of the metric

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu},$$

called the metric tensor.

Homothety: [56] The field $\mathbf{X}$ on the manifold S is a homothety if the Lie derivative of the metric tensor $g_{\mu\nu}$ of S with respect to the field $\mathbf{X}$ is equal to constant time $g_{\mu\nu}$, that is

$$\mathcal{L}_{\mathbf{X}}(g_{\mu\nu}) := g_{\mu\nu,\lambda}\mathbf{X}^{\lambda} + g_{\mu\lambda}\mathbf{X}^{\lambda}{}_{\nu} + g_{\nu\lambda}\mathbf{X}^{\lambda}{}_{\mu} = cg_{\mu\nu}, \quad (\mu, \nu, \lambda = 0, 1, 2, 3).$$

where c is a constant.

1.4 Lie Point Transformation

1.4.1 Particular Cases of Lie Point Transformations

(i): Translation: The transformation of the form

$$\tilde{x} = x + a, \quad \tilde{y} = y + b,$$

is called translation in x and y axis, these transformations form groups called Lie groups of point transformation.

(ii): Rotation: The transformation of the form

$$\tilde{x} = x\cos(t) - y\sin(t), \quad \tilde{y} = x\sin(t) + y\cos(t),$$

is called rotation. The area of geometric objects remain invariant under translation and rotation.

(iii): Scaling: The transformation

$$\tilde{x} = e^{a}x, \quad \tilde{y} = e^{b}y,$$

is called scaling, similarity transformation or dilation. This type of transformation expands or contracts the geometrical objects. The expansion or contruction is said to be uniform if $a = b$ and non-uniform otherwise.

Definition: Two geometrical figures are said to be similar if one is obtained from the other by translation, rotation or scaling transformation on the plane [56].

Example 3: Any rectangle $\{0 \leq x \leq a, \quad 0 \leq y \leq b\}$ is similar to the unit square: $\{0 \leq x \leq 1, \quad 0 \leq y \leq 1\}$. We can see that the stretching

$$\tilde{x} = \frac{x}{a}, \quad \tilde{y} = \frac{y}{b}, \tag{1.4.1}$$

converts the rectangular region $\{0 \leq x \leq a, \quad 0 \leq y \leq b\}$ into the unit square $\{0 \leq \tilde{x} \leq 1, \quad 0 \leq \tilde{y} \leq 1\}$. Using the transformation given by equations (1.4.1) one can find the relation between the areas of the two figures as

$$\tilde{x}\tilde{y} = \frac{xy}{ab} \Rightarrow 1 = \frac{xy}{ab} \Rightarrow ab = xy.$$

Similarly the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

can be transformed into the unit circle

$$\tilde{x}^2 + \tilde{y}^2 = 1,$$

by the similarity transformation given by equations (1.4.1) and their areas are related by

$$\tilde{A} = \frac{A}{ab} \Rightarrow \quad ab\tilde{A} = A \Rightarrow \quad ab\pi = A,$$

where $\tilde{A} = \pi$ is the area of the unit circle and A is the area of the ellipse.

1.4.2 General Lie Point Transformation

For coordinates (x, y) where x is independent and y is dependent variables the transformation of the form

$$\widetilde{x} = \widetilde{x}(x, y) = x + \epsilon\xi(x, y),$$
$$\widetilde{y} = \widetilde{y}(x, y) = y + \epsilon\eta(x, y),$$

is called a Lie point transformation, theses transformation can be extend to the order of differential equation. For example if we have a differential equation of the form

$$f(x, y, y^{'}, y^{''}, ..., y^k) = 0,$$

where y^k denotes k^{th} derivative with respect to x, then the Lie point transformation takes the form

$$\begin{aligned}
&\widetilde{x} = \widetilde{x}(x, y) = x + \epsilon\xi(x, y),\\
&\widetilde{y} = \widetilde{y}(x, y) = y(x, y) + \epsilon\eta(x, y),\\
&\widetilde{y}^{'} = \widetilde{y}^{'}(x, y, y^{'}) = y^{'}(x, y, y^{'}) + \epsilon\eta^1(x, y, y^{'}),\\
&\vdots\\
&\widetilde{y}^k = \widetilde{y}^k(x, y, y^{'}, ..., y^k) = y^k(x, y, y^{'}, ..., y^k) + \epsilon\eta^k(x, y, y^{'}, ..., y^k).
\end{aligned}$$

If we have n independent variables $\mathbf{x} = (x^1, x^2, ..., x^n)$ and m dependent variables $\mathbf{y} = (y^1, y^2, ..., y^m)$ then the above transformation takes the form

$$
\begin{aligned}
&\widetilde{x}^i = \widetilde{x}^i(x, y) = x^i + \epsilon\xi^i(x, y),\\
&\widetilde{y}^j = \widetilde{y}^j(x, y) = y^j + \epsilon\eta^j(x, y),\\
&\widetilde{y}^j_{i_1} = \widetilde{y}^j_{i_1}(x, y, y_{i_1}) = y^j_{i_1}(x, y, y_{i_1}) + \epsilon\eta^j_{i_1}(x, y, y_{i_1}),\\
&\vdots\\
&\widetilde{y}^j_{i_1,i_2,...,i_k} = \widetilde{y}^j_{i_1,i_2,...,i_k}(x, y, y_{i_1}, ..., y_{i_1,i_2,...,i_k}) =\\
&y^j_{i_1,i_2,...,i_k}(x, y, y_{i_1}, ..., y_{i_1,i_2,...,i_k}) + \epsilon\eta^j_{i_1,i_2,...,i_k}(x, y, y_{i_1}, ..., y_{i_1,i_2,...,i_k}),
\end{aligned}
$$

where the subscripts denote derivatives and superscripts denote coordinations.

1.5 Jet Space and Lie Symmetry Generators for Differential Equations

1.5.1 Jet Space and Lie Symmetry Generator for Ordinary Differential Equations (ODEs)

If x is the independent variable and y the dependent variable then the space underlying it is $X \times Y \cong \mathbb{R}^2$ with coordinate (x, y), the corresponding jet space of order n for nth order differential equation

$$
f(x, y, y', y'', ...y^n) = 0, \tag{1.5.1}
$$

is $X \times Y^{n+1} \cong \mathbb{R}^{n+2}$ with coordinate $(x, y, y', y'', ...y^n)$, where y^n denote the n^{th} derivative of y with respective x.

The Lie symmetry generator for the space $X \times Y = \mathbb{R}^2$ is

$$
\mathbf{X} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y},
$$

while the corresponding Lie generator for the space $X \times Y^{n+1} \cong \mathbb{R}^{n+2}$ for any nth order ODE will be

$$
\mathbf{X}^{[\mathbf{n}]} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \eta_x(x, y, y')\frac{\partial}{\partial y'} + \eta_{x,x}(x, y, y', y'')\frac{\partial}{\partial y''} + .. + \eta_{\underbrace{x, x, .., x}_{n-times}}(x, y', .., y^n)\frac{\partial}{\partial y^n}, \tag{1.5.2}
$$

where

$$\eta_x(x, y.y^{'}) = \frac{d}{dx}\eta(x, y) - y^{'}\frac{d}{dx}\xi(x, y),$$
$$\eta_{x,x}(x, y, y^{'}, y^{''}) = \frac{d}{dx}\eta_x(x, y, y^{'}) - y^{''}\frac{d}{dx}\xi(x, y),$$
$$\vdots$$
$$\eta_{\underbrace{x, x, .., x}_{n-times}}(x, y, y^{'}, .., y^n) = \frac{d}{dx}\eta_{\underbrace{x, x, .., x}_{(n-1)-times}}(x, y, y^{'}, .., y^{n-1}) - y^n\frac{d}{dx}\xi(x, y),$$

and the total derivative operator is

$$\frac{d}{dx} = \frac{\partial}{\partial x} + y^{'}\frac{\partial}{\partial y} + y^{''} + ... + y^n\frac{\partial}{\partial y^{n-1}}.$$

Example 4: Consider the example

$$y^{''}(x) + y(x) = 0. \tag{1.5.3}$$

The jet space for this differential equation is $X \times Y^3 = \mathbb{R}^4$ with jet coordinate $(x, y, y^{'}, y^{''})$. The Lie symmetry generator takes the form

$$\mathbf{X} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y},$$

and the corresponding second order extended generator is

$$\mathbf{X}^{[\mathbf{2}]} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \eta_x(x, y, y^{'})\frac{\partial}{\partial y'} + \eta_{xx}(x, y, y^{'}, y^{''})\frac{\partial}{\partial y''}. \tag{1.5.4}$$

Apply the generator given by equation (1.5.4) on the ODE given by equation(1.5.3) and using the values of η_x and η_{xx} in terms of η we have the following system of determining equations,

$$\begin{aligned}
&\xi_{yy} = 0, \quad \xi_{xxy} + \xi_y = 0, \quad \xi_{xxx} - 3\xi_{xy}y + 4\xi_x = 0,\\
&\eta_{yy} - 2\xi_{xy} = 0, \quad \eta_{xy} + 3\xi_y y - \xi_{xx} = 0,\\
&\eta_{xx} - \eta_y y + 2\xi_x y + \eta = 0.
\end{aligned} \tag{1.5.5}$$

Solution of system given in equations (1.5.5) is

$$\begin{aligned}
\xi(x, y) &= c_1 y\sin(x) + c_2 y\cos(x) + c_4\sin(2x) + c_5\cos(2x) + c_3,\\
\eta(x, y) &= c_1 y^2\cos(x) - c_2 y^2\sin(x) + c_4 y\cos(2x) - c_5 y\sin(2x) + c_6 y + c_7\sin(x) + c_8\cos(x).
\end{aligned} \tag{1.5.6}$$

This solution represents eight parameter Lie group and hence forms eight dimensional Lie algebra. The symmetry generators are

$$\begin{aligned}
&\mathbf{X_1} = y\sin(x)\frac{\partial}{\partial x} + y^2\cos(x)\frac{\partial}{\partial y}, \quad \mathbf{X_2} = y\cos(x)\frac{\partial}{\partial x} - y^2\sin(x)\frac{\partial}{\partial y},\\
&\mathbf{X_3} = \sin(2x)\frac{\partial}{\partial x} + y\cos(2x)\frac{\partial}{\partial y}, \quad \mathbf{X_4} = \cos(2x)\frac{\partial}{\partial x} - y\sin(2x)\frac{\partial}{\partial y},\\
&\mathbf{X_5} = \cos(x)\frac{\partial}{\partial y}, \quad \mathbf{X_6} = \sin(x)\frac{\partial}{\partial y}, \mathbf{X_7} = y\frac{\partial}{\partial y}, \quad \mathbf{X_8} = \frac{\partial}{\partial x}.
\end{aligned} \tag{1.5.7}$$

The Lie algebra is

$$\begin{aligned}
&[\mathbf{X_1}, \mathbf{X_3}] = -\mathbf{X_1}, \quad [\mathbf{X_1}, \mathbf{X_3}] = -\mathbf{X_2}, \quad [\mathbf{X_1}, \mathbf{X_5}] = -\frac{1}{2}\mathbf{X_3} - \frac{3}{2}\mathbf{X_7}\\
&[\mathbf{X_1}, \mathbf{X_6}] = \frac{1}{2}\mathbf{X_4} - \frac{1}{2}\mathbf{X_8}, \quad [\mathbf{X_1}, \mathbf{X_7}] = -\mathbf{X_1}, \quad [\mathbf{X_1}, \mathbf{X_8}] = -\mathbf{X_2},\\
&[\mathbf{X_2}, \mathbf{X_3}] = \mathbf{X_2}, \quad [\mathbf{X_2}, \mathbf{X_4}] = -\mathbf{X_1}, \quad [\mathbf{X_2}, \mathbf{X_5}] = -\frac{1}{2}\mathbf{X_4} - \frac{1}{2}\mathbf{X_8},\\
&[\mathbf{X_2}, \mathbf{X_6}] = -\frac{3}{2}\mathbf{X_7} - \frac{1}{2}\mathbf{X_3}, \quad [\mathbf{X_2}, \mathbf{X_7}] = \mathbf{X_2}, \quad [\mathbf{X_2}, \mathbf{X_8}] = -\mathbf{X_1},\\
&[\mathbf{X_3}, \mathbf{X_4}] = -\mathbf{X_8}, \quad [\mathbf{X_3}, \mathbf{X_5}] = -\mathbf{X_5}, \quad [\mathbf{X_3}, \mathbf{X_6}] = \mathbf{X_6},\\
&[\mathbf{X_3}, \mathbf{X_8}] = -2\mathbf{X_4}, \quad [\mathbf{X_4}, \quad \mathbf{X_5}] = \mathbf{X_6}, [\mathbf{X_4}, \mathbf{X_6}] = \mathbf{X_5},\\
&[\mathbf{X_4}, \mathbf{X_8}] = 2\mathbf{X_3}, \quad [\mathbf{X_5}, \mathbf{X_7}] = \mathbf{X_5}, \quad [\mathbf{X_5}, \mathbf{X_8}] = \mathbf{X_6},\\
&[\mathbf{X_6}, \mathbf{X_7}] = \mathbf{X_6}, \quad [\mathbf{X_6}, \mathbf{X_8}] = \mathbf{X_5}, \quad [\mathbf{X_i}, \mathbf{X_j}] = 0, \quad \textit{otherwise.}
\end{aligned}$$

The corresponding one parameter Lie groups are (a is the parameter)

$$\begin{aligned}
G_1: &\quad \left[x + ay\sin(x), \quad \frac{y}{1 - ay\cos(x)}\right],\\
G_2: &\quad \left[x + ay\cos(x), \quad \frac{y}{1 + ay\sin(x)}\right],\\
G_3: &\quad [\arctan(\tan(x).\exp(2a)), \quad y\exp(a\cos(2x))],\\
G_4: &\quad \left[\arctan(\tan(x - \frac{\pi}{4})\exp(2a)) + \frac{\pi}{4}, \quad y\exp(-a\sin(2x))\right],\\
G_5: &\quad [x, \quad y + a\cos(x)],\\
G_6: &\quad [x, \quad y + a\sin(x)],\\
G_7: &\quad [x, \quad \exp(a)y],\\
G_8: &\quad [x + a, \quad y].
\end{aligned}$$

1.5.2 Jet Space and Lie Symmetry Generator for Partial Differential Equations (PDEs)

For $\mathbf{x} = (x^1, x^2, ..., x^n)$ independent and $\mathbf{u} = (u^1, u^2, .., u^m)$ dependent variables the Euclidian space is $\mathbf{x} \times \mathbf{u} = \mathbb{R}^{n+m}$ and, in coordinate notation it is $(\mathbf{x}, \mathbf{u})$ and its corresponding k^{th} order jet space is $\mathbf{x} \times \mathbf{u}_{i_1 i_2,...i_k}$ with coordinate $(x, u, u_{i_1}, u_{i_1 i_2}, ..., u_{i_1 i_2 ... i_k})$, where the subscript denotes the derivatives. The k^{th} order partial differential equation will be of the form

$$f(x, u, u_{i_1}, u_{i_1 i_2}, ..., u_{i_1 i_2 ... i_k}) = 0.$$

The Lie generator for this PDE is

$$\mathbf{X} = \xi^i \frac{\partial}{\partial x^i} + \eta^j \frac{\partial}{\partial u^j},$$

and its k^{th} order prolongation is

$$\mathbf{X}^{[\mathbf{k}]} = \xi^i \frac{\partial}{\partial x^i} + \eta^j \frac{\partial}{\partial u^j} + \eta^j_{i_1} \frac{\partial}{\partial u^j_{i_1}} + \eta^j_{i_1 i_2} \frac{\partial}{\partial u^j_{i_1 i_2}} + ... + \eta^j_{i_1 ... i_k} \frac{\partial}{\partial u^j_{i_1 ... i_k}}.$$

Example 5: Consider the following heat equation

$$\phi_t(t, x) - \phi_{xx}(t, x) = 0. \tag{1.5.8}$$

It is second order linear PDE, its solution space is $\mathbf{X} \times \Phi$ with coordinates (t, x, ϕ) and the corresponding jet space is $\mathbf{X} \times \Phi^2$ with jet coordinates $(t, x, \phi, \phi_t, \phi_x, \phi_{tt}, \phi_{tx}, \phi_{xx})$.

The symmetry generator for the solution space is

$$\mathbf{X} = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial \phi},$$

and the second order prolonged generator is

$$\mathbf{X}^{[\mathbf{2}]} = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial \phi} + \eta_t \frac{\partial}{\partial \phi_t} + \eta_x \frac{\partial}{\partial \phi_x} + \eta_{tt} \frac{\partial}{\partial \phi_{tt}} + \eta_{tx} \frac{\partial}{\partial \phi_{tx}} + \eta_{xx} \frac{\partial}{\partial \phi_{xx}}. \tag{1.5.9}$$

Apply the generator given by equation (1.5.9) on equation (1.5.8) and inserting the values of η_t, η_{xx} in term of η and ξ^i we get the following system of determining equations

$$\begin{aligned}
&\xi^2_\phi = 0, \quad \xi^2_x = 0, \quad \xi^2_{\phi\phi} = 0, \quad \xi^1_\phi - 2\xi^2_{x\phi} - 3\xi^1_\phi = 0, \quad \xi^1_{\phi\phi} = 0, \\
&2\xi^1_x - \xi^2_t + \xi^2_{xx} - \eta_\phi = 0, \quad 2\xi^2_{x\phi} - \eta_{\phi\phi} = 0, \\
&\xi^1_{xx} - \xi^1_t - 2\eta_{x\phi} = 0, \quad \eta_t - \eta_{xx} = 0.
\end{aligned} \tag{1.5.10}$$

The solution of this system is

$$\begin{aligned}
\xi^1 &= 2c_1 t + 4c_3 t^2 + c_4, \\
\xi^2 &= c_1 x + 2c_2 t + 4c_3 x t + c_5, \\
\eta &= -c_2 x\phi - 2c_3 t\phi - c_3\phi x^2 + c_6\phi + \beta(t,x).
\end{aligned} \tag{1.5.11}$$

As we have an arbitrary function $\beta(t,x)$ in the solution so the algebra is infinite dimensional here. The Lie symmetry generators are

$$\begin{aligned}
&\mathbf{X_1} = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \quad \mathbf{X_2} = 2t\frac{\partial}{\partial x} - x\phi\frac{\partial}{\partial \phi}, \quad \mathbf{X_3} = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - \phi(2t + x^2)\frac{\partial}{\partial \phi}, \\
&\mathbf{X_4} = \frac{\partial}{\partial t}, \quad \mathbf{X_5} = \frac{\partial}{\partial x}, \quad \mathbf{X_6} = \phi\frac{\partial}{\partial \phi}, \quad \mathbf{X}_\beta = \beta(t,x)\frac{\partial}{\partial \phi}.
\end{aligned} \tag{1.5.12}$$

The corresponding Lie groups are

$$\begin{aligned}
G_1 :& \quad [\exp(2a)t, \quad \exp(a)x, \quad \phi], \\
G_2 :& \quad \left[t, \quad x + 2at, \quad \phi\exp(-ax - a^2 t)\right], \\
G_3 :& \quad \left[\frac{t}{1-4at}, \quad \frac{x}{1-4ax}, \quad \phi\sqrt{1-4at}\exp\left(\frac{-ax^2}{1-4at}\right)\right], \\
G_4 :& \quad [t + a, \quad x, \quad \phi], \\
G_5 :& \quad [t, \quad x + a, \quad \phi], \\
G_6 :& \quad [t, \quad x, \quad \exp(a)\phi], \\
G_\beta :& \quad [t, \quad x, \quad \phi + a\beta(t,x)].
\end{aligned} \tag{1.5.13}$$

1.6 Symmetries More General than Point Symmetries

1.6.1 Contact Transformation and Lie-Backlund Transformation

Contact Transformation: The transformation in which ξ and η are function of $x, y, y^{'}$ are called contact transformations $[8, 10, 11, 36]$. Consider the differential equation

$$y^{''}(x) + y^{'}(x) + y^2(x) = 0.$$

This differential equation admits the only symmetry

$$\mathbf{X} = \frac{\partial}{\partial x}.$$

Now taking the transformation

$$\widetilde{y} = y^{'} = \nu, \quad \widetilde{x} = y = \mu,$$

the given differential equation takes the form

$$(\nu + 1)\frac{d\nu}{d\mu} + \mu^2 = 0.$$

This is first order differential equation, so it has infinitely many symmetries of the form

$$\mathbf{X_1} = m(\mu,\nu)\frac{\partial}{\partial \mu} + n(\mu,\nu)\frac{\partial}{\partial \nu}.$$

Transforming back this symmetry generator into the original coordinate (x, y) we have

$$\begin{aligned}
&\mathbf{X_1} = \mathbf{X_1}(x)\frac{\partial}{\partial x} + \mathbf{X_1}(y)\frac{\partial}{\partial y} + \mathbf{X_1}(y^{'})\frac{\partial}{\partial y^{'}} \\
&= m(\mu,\nu)\frac{\partial}{\partial \mu} + n(\mu,\nu)\frac{\partial}{\partial \nu} \\
&= m(y,y^{'})\frac{\partial}{\partial y} + n(y,y^{'})\frac{\partial}{\partial y^{'}}.
\end{aligned}$$

Now we see that the symmetry generator $\mathbf{X_1}$ is not a Lie point symmetry generator for the point transformation because the coefficient of $\frac{\partial}{\partial y}$ depends upon $y^{'}$, and similarly the transformation we have taken here is not the point transformation, that is why the symmetry $\mathbf{X_1}$ is not show up for the given differential equation. This type of transformation which depends upon x, y and the first derivative of y is called a contact transformation.

Lie-Bäcklund Transformation: This is the most general form of the transformation in which the transformation depends upon n independent variables $x = (x^1, x^2, ..., x^n)$, m dependent variables $u = (u^1, u^2, ..., u^m)$ and up to k^{th} derivative of the dependent variables, that is

$$\begin{aligned}
&\widetilde{x} = \widetilde{x}(x, u, u_{i_1}, u_{i_1 i_2}, ...u_{i_1 i_2 ... i_k}), \\
&\widetilde{u} = \widetilde{u}(x, u, u_{i_1}, u_{i_1 i_2}....u_{i_1 i_2 ... i_k}), \\
&\widetilde{u}_{i_1} = \widetilde{u}_{i_1}(x, u, u_{i_1}, u_{i_1 i_2}, ...u_{i_1 i_2 ... i_k}), \\
&\vdots \\
&\widetilde{u}_{i_1 i_2 ... i_k} = \widetilde{u}_{i_1 i_2 ... i_k}(x, u, u_{i_1}, u_{i_1 i_2}, ...u_{i_1 i_2 ... i_k}).
\end{aligned} \tag{1.6.1}$$

This type of transformation is also called higher order tangent transformation.

1.6.2 Approximate Lie Group and Lie Symmetry Generator

Physical problems some times admit approximation. For example we ignore the air friction in free fall. Similarly the simple pendulum is effected by the air friction which dies away the motion of pendulum, the simple harmonic motion of a body attached to a spring is reduce by the friction of the surface on which it moves. These small perturbations in physical systems (here of the air resistance and friction between the spring and the surface on which it moves) are very sensitive to the exact Lie group theoretic approach to the solution of differential equations [31]. Consequently the application of the Lie group technique to the solution of differential equations reduces much in such physical problems. Fortunately an approximate Lie group technique were developed [4,5,64] and used to reduce the instability of the Lie group theoretic method to the solution of differential equations.

Definition: An approximate transformation of order p in $\mathbb{R}^n$ can be written as [42]

$$x^i \to \tilde{x}^i \approx \tilde{x}^i_0(x^i, a) + \epsilon \tilde{x}^i_1(x^i, a) + ... + \epsilon^p \tilde{x}^i_p(x^i, a), \tag{1.6.2}$$

which obey the initial conditions

$$\tilde{x}^i_j|_{a=0} \approx x^i_j, \forall i.$$

Approximate symmetry group generator of order p: The generator of the approximate transformation given in equation (1.6.2) is of the form

$$\mathbf{X} = \xi^i(\mathbf{x}, \epsilon)\frac{\partial}{\partial x^i}, \tag{1.6.3}$$

such that

$$\xi^i(\mathbf{x}, \epsilon) \approx \xi^i_0(\mathbf{x}) + \epsilon\xi^i_1(\mathbf{x}) + ...\epsilon^p\xi^i_p(\mathbf{x}), \quad \xi^i_j(x) = \frac{\partial}{\partial a}\tilde{x}^i_j|_{a=0}, \tag{1.6.4}$$

then the generator given in equation (1.6.3) takes the form

$$\mathbf{X} = (\xi^i_0(\mathbf{x}) + \epsilon\xi^i_1(\mathbf{x}) + ...\epsilon^p\xi^i_p(\mathbf{x}))\frac{\partial}{\partial x^i}. \tag{1.6.5}$$

First order approximation: The symmetry generator of the form

$$\mathbf{X} = \xi^i\frac{\partial}{\partial x^i}, \tag{1.6.6}$$

is said to be of the first order if $\xi^i = \xi_0^i + \epsilon\xi_1^i$, where ϵ is a small arbitrary parameter. The generator in equation (1.6.6) splits into two parts as

$$\mathbf{X_0} = \xi_0^i \frac{\partial}{\partial x^i}, \quad \mathbf{X_1} = \xi_1^i \frac{\partial}{\partial x^i}, \tag{1.6.7}$$

where $\mathbf{X_0}$ is the exact and $\mathbf{X_1}$ is the approximate part of the symmetry generator given in equation (1.6.6).

The corresponding approximate transformation group of point x into $\tilde{x}$ is

$$x^i \to \tilde{x}^i(x^i, a) = \tilde{x}_0^i(x^i, a) + \epsilon\tilde{x}_1^i(x^i, a). \tag{1.6.8}$$

Example 6: The one dimensional symmetry generator

$$\mathbf{X} = (x^2 + \epsilon x)\frac{\partial}{\partial x}, \tag{1.6.9}$$

splits into two parts as

$$\mathbf{X_0} = x^2 \frac{\partial}{\partial x}, \quad \mathbf{X_1} = x\frac{\partial}{\partial x},$$

where $\mathbf{X_0}$ is the exact and $\mathbf{X_1}$ is the approximate symmetry. Here $\xi_0 = x^2$ and $\xi_1 = x$.

The corresponding approximate Lie equations are

$$\begin{aligned} \frac{d\tilde{x}_0}{da} &= x^2, \quad \tilde{x}_0|_{a=0} = x, \\ \frac{d\tilde{x}_1}{da} &= \tilde{x}_0, \quad \tilde{x}_1|_{a=0} = 0. \end{aligned}$$

The solution to this system is

$$\tilde{x}_0 = ax^2 + x, \quad \tilde{x}_1 = \frac{a^2x^2}{2} + ax.$$

The approximate group is

$$\tilde{x} = ax^2 + x + \epsilon\left(\frac{a^2x^2}{2} + ax\right).$$

Example 7: Now consider a two dimensional symmetry generator

$$\mathbf{X} = (1 + \epsilon x)\frac{\partial}{\partial x} + \epsilon y\frac{\partial}{\partial y},$$

which can be splits into two parts as

$$\mathbf{X_0} = \frac{\partial}{\partial x}, \quad \mathbf{X_1} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

where $\mathbf{X_0}$ is the exact and $\mathbf{X}_1$ is the approximate symmetry. The corresponding approximate Lie differential equations are

$$\frac{d\tilde{x}_0}{da} = 1, \quad \frac{d\tilde{y}_0}{da} = 0, \quad \tilde{x}_0|_{a=0} = x, \quad \tilde{y}_0|_{a=0} = y,$$
$$\frac{d\tilde{x}_1}{da} = \tilde{x}_0, \quad \frac{d\tilde{y}_0}{da} = \tilde{y}_0, \quad \tilde{x}_1|_{a=0} = 0, \quad \tilde{y}_1|_{a=0} = 0.$$

The solution of this system is

$$\tilde{x}_0 = x + a, \quad \tilde{x}_1 = ax + \frac{a^2}{2}, \quad \tilde{y}_0 = y, \quad \tilde{y}_1 = ay,$$

the approximate group corresponding to the above solution is

$$\tilde{x} = x + a + \epsilon\left(ax + \frac{a^2}{2}\right), \quad \tilde{y} = y + \epsilon(ay).$$

Example 8: Consider the following second order differential equation

$$y^{''} = 0. \tag{1.6.10}$$

Using the symmetry generator

$$\mathbf{X}^{[\mathbf{2}]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial x} + \eta^1(x,y,y^{'})\frac{\partial}{\partial y^{'}} + \eta^2(x,y,y^{'},y^{''})\frac{\partial}{\partial y^{''}}.$$

We have the following system of determining partial differential equations

$$\xi_{yy} = 0, \quad \xi_{xx} = 2\eta_{xy}, \quad \xi_{xy} = \frac{1}{2}\eta_{yy}, \quad \eta_{xx} = 0.$$

The solution of this system is

$$\xi = c_1 + c_2x + c_3y + c_4x^2 + c_5\frac{xy}{2},$$
$$\eta = c_4xy + c_5\frac{y^2}{2} + c_6x + c_7y + c_8.$$

The symmetry generators are

$$\mathbf{X_1} = \frac{\partial}{\partial x}, \quad \mathbf{X_2} = x\frac{\partial}{\partial x}, \quad \mathbf{X_3} = y\frac{\partial}{\partial x}, \quad \mathbf{X_4} = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},$$
$$\mathbf{X_5} = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \quad \mathbf{X_6} = y\frac{\partial}{\partial y}, \quad \mathbf{X_7} = x\frac{\partial}{\partial y}, \quad \mathbf{X_8} = \frac{\partial}{\partial y}.$$

Perturbing the differential equation given in equation (1.6.10) as

$$y^{''} + \epsilon y = 0, \tag{1.6.11}$$

where ϵ is a small parameter and applying the approximate symmetry

$$\mathbf{X}^{[2]} = (\xi_0 + \epsilon\xi_1)\frac{\partial}{\partial x} + (\eta_0 + \epsilon\eta_1)\frac{\partial}{\partial y} + (\eta_0^{'} + \epsilon\eta_1^{'})\frac{\partial}{\partial y^{'}} + (\eta_0^{''} + \epsilon\eta_1^{''})\frac{\partial}{\partial y^{''}}, \qquad (1.6.12)$$

on differential equation (1.6.11) getting the terms of order one in ϵ and solving the system we have the following solution,

$$\begin{aligned}
&\xi_0 + \epsilon\xi_1 = c_1(1 - 2\epsilon x^2) + c_2\left(x - \epsilon\frac{2x^3}{3}\right) + c_3\left(y - \epsilon\frac{x^2 y}{2}\right) + c_4\left(x^2 - \epsilon\frac{x^4}{2}\right) + \\
&c_5\left(xy - \epsilon\frac{x^3 y}{6}\right) + c_6\epsilon + c_7\epsilon x + c_8\epsilon xy + c_9\epsilon 2x + c_{10}\epsilon y + c_{11}\epsilon x^2, \\
&\eta_0 + \epsilon\eta_1 = -2c_1\epsilon xy + c_2\epsilon(y - yx^2) - c_3\epsilon xy^2 + c_4\left(yx - \epsilon\frac{2yx^3}{3}\right) + c_5\left(y^2 - \epsilon\frac{y^2x^2}{2}\right) + \\
&c_8\epsilon y^2 + c_9\epsilon y + c_{11}\epsilon xy + c_{12}\left(x - \epsilon\frac{x^3}{6}\right) + c_{13}y + c_{14}\left(1 - \epsilon\frac{x^2}{2}\right) + c_{15}\epsilon x + c_{16}\epsilon.
\end{aligned}$$

The corresponding symmetry generators are

$$\begin{aligned}
&\mathbf{X_1} = (1 - 2\epsilon x^2)\frac{\partial}{\partial x} - 2\epsilon xy\frac{\partial}{\partial y}, \quad \mathbf{X_2} = \left(x - \epsilon\frac{2x^3}{3}\right)\frac{\partial}{\partial x} + \epsilon(y - x^2y)\frac{\partial}{\partial y}, \\
&\mathbf{X_3} = \left(y - \epsilon\frac{x^2y}{2}\right)\frac{\partial}{\partial x} - \epsilon xy^2\frac{\partial}{\partial y}, \quad \mathbf{X_4} = \left(x^2 - \epsilon\frac{x^4}{2}\right)\frac{\partial}{\partial x} - \left(xy - \epsilon\frac{2x^3y}{3}\right)\frac{\partial}{\partial y}, \\
&\mathbf{X_5} = \left(xy - \epsilon\frac{x^3y}{6}\right)\frac{\partial}{\partial x} - \left(y^2 - \epsilon\frac{x^2y^2}{2}\right)\frac{\partial}{\partial y}, \quad \mathbf{X_6} = \epsilon\frac{\partial}{\partial x,}, \quad \mathbf{X_7} = \epsilon x\frac{\partial}{\partial x}, \\
&\mathbf{X_8} = \epsilon\left(xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right), \quad \mathbf{X_9} = \epsilon\left(2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \\
&\mathbf{X_{10}} = \epsilon\left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\right), \quad \mathbf{X_{11}} = \epsilon y\frac{\partial}{\partial x}, \quad \mathbf{X_{12}} = \epsilon\left(x - \frac{x^3}{6}\right)\frac{\partial}{\partial x}, \\
&\mathbf{X_{13}} = y\frac{\partial}{\partial y}, \quad \mathbf{X_{14}} = \left(1 - \epsilon\frac{x^2}{2}\right)\frac{\partial}{\partial y}, \quad \mathbf{X_{15}} = \epsilon x\frac{\partial}{\partial y}, \quad \mathbf{X_{16}} = \epsilon\frac{\partial}{\partial y}.
\end{aligned}$$

1.7 Plan of the Thesis

This thesis is organized in the following way. A brief introduction to Euler-Lagrange equations and Noether variational problem is given in Chapter 2. We begin calculation from an action of a first order Lagrangian for relativistic field theories, where $\phi^i(x^\mu)$, $i = 1, 2, 3..., N$, are functions, which play the role of dependent variables and x^μ, $\mu = 0, 1, 2, 3$ are the independent variables. We derive the Euler-Lagrange equations and the corresponding conserved quantities for the action, and generalize this calculation for n independent and m dependent variables and Lagrangian of order k.

The Noether symmetries, using the arc length minimizing Lagrangian of plane symmetric static spacetimes are given in Chapter 3. An introduction to Noether symmetry equation and metric of general plane symmetric static spacetimes are given, and the remainder of the chapter consists of many sections. Each section contains Noether symmetries, metric of the spacetimes and the corresponding first integrals.

In Chapter 4 approximate Noether symmetries of the arc length minimizing Lagrangian of time conformal plane symmetric spacetimes are presented. This chapter consists of four sections. In section one, the definition of approximate Noether symmetry and perturbed Lagrangian for the time conformal plane symmetric spacetimes along with a system of 19 PDEs are given. The remaining three sections consist of those cases where the approximate symmetry(ies) exist(s) and list all those time conformal plane symmetric spacetimes where the approximate Noether symmetries exist. The approximate first integrals are also given in this chapter.

The Noether symmetries of the arc length minimizing Lagrangian of cylindrically symmetric static spacetimes are given in Chapter 5. Different sections consist of different numbers of Noether symmetries along with the corresponding cylindrically symmetric static spacetimes and first integrals.

Chapter 6 consists of the complete classification of spherically symmetric static spacetimes according to Noether symmetries.

The conclusion of the thesis, discussion on some new cases of plane symmetric static, spherically symmetric static spacetimes and vacuum solutions are given in Chapter 7.

Chapter 2

Preliminaries

2.1 Introduction

The symmetries of a variational problem are called Noether symmetries. A variational problem describes a physical system and can be written in an integral form which is called the action of the problem. For example consider length of a curve of function $f(t)$, from a point $(a, f(a))$ to another point $(b, f(b))$ that is

$$S = \int_a^b \sqrt{1 + \dot{f(t)}^2}dt, \tag{2.1.1}$$

where " $\cdot$ " denotes differentiation with respect to t. For minimum value of S, $f(t)$ must be a straight line, which is the extremal value of $f(t)$. To show that $f(t)$ is a straight line we shift $f(t)$ from its minimum value by $\epsilon v(t)$ that is

$$f(t) \to f(t) + \epsilon v(t), \tag{2.1.2}$$

where $v(t)$ is arbitrary function satisfying $v(a) = v(b) = 0$ and ϵ is an arbitrary small parameter. The integral given in equation (2.1.1) takes the form

$$S_v = \int_a^b \sqrt{1 + (\dot{f}(t) + \epsilon\dot{v}(t))^2}dt. \tag{2.1.3}$$

Differentiating with respect to ϵ we have

$$\begin{aligned} \frac{d}{d\epsilon}S_v &= \frac{d}{d\epsilon}\int_a^b \sqrt{1+(\dot{f}(t)+\epsilon\dot{v}(t))^2}dt, \\ &= \int_a^b \frac{\dot{f}+\epsilon\dot{v}}{\sqrt{1+(\dot{f}(t)+\epsilon\dot{v}(t))^2}}\dot{v}|_{\epsilon=0}dt, \\ &= \int_a^b \frac{\dot{f}(t)}{\sqrt{1+\dot{f}(t)^2}}\dot{v}dt. \end{aligned} \tag{2.1.4}$$

Integrating the right hand side with respect to t we have

$$\begin{aligned} \frac{d}{d\epsilon}S_v &= \frac{\dot{f}}{\sqrt{1+\dot{f}(t)^2}}v|_a^b - \int_a^b \frac{d}{dt}\left[\frac{\dot{f}}{\sqrt{1+\dot{f}(t)^2}}\right]vdt, \\ &= -\int_a^b \frac{d}{dt}\left[\frac{\dot{f}}{\sqrt{1+\dot{f}(t)^2}}\right]vdt. \end{aligned} \tag{2.1.5}$$

For extremal values of S the variation vanishes which imposes condition on the function $f(t)$. That is

$$\frac{d}{d\epsilon}S_v = -\int_a^b \frac{d}{dt}\left[\frac{\dot{f}}{\sqrt{1+\dot{f}(t)^2}}\right]vdt = 0, \tag{2.1.6}$$

since $v(t)$ is arbitrary, therefore,

$$-\frac{d}{dt}\left[\frac{\dot{f}}{\sqrt{1+\dot{f}(t)^2}}\right] = 0, \tag{2.1.7}$$

which implies that $f(t)$ must be a straight line. If we take

$$\mathcal{L}(t, f(t), \dot{f}(t)) = \sqrt{1+\dot{f}^2(t)},$$

then equation (2.1.7) can be written as

$$\frac{\partial\mathcal{L}}{\partial f(t)} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{f}(t)} = 0, \tag{2.1.8}$$

which is the Euler-Lagrange equation for the action given by equation (2.1.1), where the function $\mathcal{L}$ is called Lagrangian density. This is a specific variational problem, the calculation could be extended to any general Lagrangian depending on general coordinates and derivatives of any order.

The Euler-Lagrange equations are the restriction on the variables involved in the action. These equations could be obtained from the variation in the action. The solution of these equations give the extremal values of the action. In the following sections a brief introduction to Noether variational problem and Euler-Lagrange equations are given. At the end of this chapter the definitions of first order and k^{th} order approximate Noether symmetries are given.

2.2 A Short Review of Noether Variational Problem and Euler-Lagrange Equations

Emmy Noether: Emmy Noether was born on 23 March 1882 in Erlangen, Bavaria, Germany [82]. She was named Amalie, but always called “Emmy”. Emmy Noether's father Max Noether was a recognized mathematician and a teacher at Erlangen. Her mother was Ida Kaufmann, from a well known and wealthy family of Erlangen. Both Emmy's parents were of Jewish cast and Emmy was the eldest of their four children. Emmy Noether went to the Hohere Tochter School in Erlangen from 1889 until 1897. She studied German, English, French, along with arithmetics and was given piano lessons. At this stage she wanted to become a language teacher, and after further investigation of English and French she took the examinations of the State of Bavaria and, in 1900, turned into a certificated instructor of English and French in Bavarian girls school. However, Noether never turned into a language instructor, rather she chose to take the troublesome course for a lady of that time and studied mathematics at university. Ladies were permitted to study at German institutions informally and every teacher allowed the female to attend his lecture. Noether was allowed to sit in courses at the University of Erlangen throughout 1900 to 1902. Having taken and passed the registration examination in Nurnberg in 1903, she went to the University of Gottingen. From 1903 to 1904 she attended the lectures given by Blumenthal, Hilbert, Klein and Minkowski. She got her PhD degree in mathematics under the supervision of Paul Gordan. Having finished her doctorate, the typical movement to a scholarly post might have been the habilitation. However, this route was not open to ladies so Noether stayed at Erlangen, helping her father who, especially on account of his own incapacities, was appreciative for his daughter's assistance. Noether additionally continued her research, during this period she was affected by Fischer who had succeeded

Gordan in 1911. This impact took Noether towards Hilbert's conceptual methodology to the subject. In 1915 Hilbert and Klein welcomed Noether to come back to Gottingen [82]. They convinced her to stay at Gottingen while they battled a fight to have her on the faculty of the university. In a long fight with the university administration, to permit Noether to get her habilitation there, she got the permission in 1919 after four years of struggle. Throughout this time Hilbert had permitted Noether to lecture by publicizing her courses under his own particular name. For instance a course given in the winter semester of 1916-17 shows up in the index as:

Mathematical Physics Seminar: Professor Hilbert, with the assistance of Dr. E Noether, Mondays from 4-6, no tuition.

Emmy Noether's first excellent work, when she was in Gottingen in 1915 is an outcome in theoretical physics [18, 52, 53], known as Noether's theorem, which demonstrates a relationship between symmetries in physics and conservation laws. This basic result in the general theory of relativity was praised by Albert Einstein in a letter to Hilbert when he referred to Noether's *penetrating mathematical thinking.*

Noether published this paper called "Invariant variational problem", now famous for connecting symmetries with conservation laws. This paper was presented to the Royal Society in Gottingen on July 17, 1918 by Felix Klein [58]. Her discussion begins with a variational problem and its solution. Noether's paper concerns theories that can be given in a Lagrangian formulation. We begin with basic features of such theories. For relativistic field theories, we take the Lagrangian density [19, 46]

$$\mathcal{L} = \mathcal{L}(x^\mu, \phi^i, \partial_\mu \phi^i), \tag{2.2.1}$$

where ϕ^i $(i = 1, 2..., N)$ are fields (dependent variables) depending on independent variables x^μ $(\mu = 0, 1, 2, 3)$. The action S is defined in terms of the Lagrangian density as

$$S = \int_R \mathcal{L}(x^\mu, \phi^i, \partial^\mu \phi^i) d^4x, \tag{2.2.2}$$

and the corresponding Euler-Lagrange equations take the form

$$\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} = 0. \tag{2.2.3}$$

To discuss Noether variational problem, we first need to understand in more detail the variations that we are considering. Consider one of the dependent variable, say $\phi^k =$

ϕ. The total variation in ϕ and the independent variable x^μ, gives rise to the following transformation of ϕ:

$$\delta\phi = \tilde{\phi}(\tilde{x}) - \phi(x). \tag{2.2.4}$$

Here the variation consist of two parts. The first is the variation in the function ϕ at a fixed coordinate, which is given by

$$\delta_o\phi = \tilde{\phi}(x) - \phi(x). \tag{2.2.5}$$

The second variation in ϕ is due to change in independent variable x, that is

$$\delta_x\phi = \phi(\tilde{x}) - \phi(x). \tag{2.2.6}$$

From equation (2.2.4) we have

$$\begin{aligned}
\delta\phi = \tilde{\phi}(\tilde{x}) - \phi(x) &= \delta_0\phi(\tilde{x}) + \phi(\tilde{x}) - \phi(x), \\
&= \delta_0\phi(x+\delta x) + \phi(x+\delta x) - \phi(x), \\
&= \delta_0\phi(x) + \delta x^\mu\partial_\mu\delta_0\phi + \phi(x) + \delta x^\mu\partial_\mu\phi(x) - \phi(x), \\
&= \delta_0\phi(x) + \delta x^\mu\partial_\mu\phi(x).
\end{aligned} \tag{2.2.7}$$

This important relation will be used in the solution of Noether variational problem. The operator ∂_μ commutes with δ_0 as (we need this commutation relation in the proof of Noether variational problem)

$$\partial_\mu\delta_o\phi = \partial_\mu\{\tilde{\phi}(x) - \phi(x)\} = \partial_\mu\tilde{\phi}(x) - \partial_\mu\phi(x) = \delta_o\partial_\mu\phi(x), \tag{2.2.8}$$

while the variation δ does not commute with the ordinary derivative operator ∂_μ

$$\begin{aligned}
\partial_\mu\delta\phi(x) &= \partial_\mu\{\tilde{\phi}(\tilde{x}) - \phi(x)\}, \\
&= \partial_\mu\tilde{\phi}(\tilde{x}) - \partial_\mu\phi(x) = \frac{\partial\tilde{x}^\nu}{\partial x^\mu}\partial_{\tilde{\nu}}\tilde{\phi}(\tilde{x}) - \partial_\mu\phi(x).
\end{aligned} \tag{2.2.9}$$

This reduces to

$$\partial_\mu\delta\phi(x) = \delta\partial_\mu\phi(x), \tag{2.2.10}$$

if and only if

$$\frac{\partial\tilde{x}^\nu}{\partial x^\mu} = \delta^\nu_\mu, \tag{2.2.11}$$

and that is possible only when x^μ are fixed.

2.2.1 Noether Variational Problem

For Noether variational problem we take variations in the dependent, independent, and the derivatives of the dependent variables [17, 68] that is

$$\begin{aligned}
&x^\mu \to \tilde{x}^\mu(x^\mu, \phi_i) = x^\mu + \epsilon\xi^\mu(x^\mu, \phi^i) + ..., \\
&\phi^i(x^\mu) \to \tilde{\phi}^i(x^\mu, \phi^i) = \phi^i(x^\mu) + \epsilon\eta^i(x^\mu, \phi^i)..., \\
&\partial_\mu\phi^i(x^\mu) \to \partial_\mu\tilde{\phi}^i(x^\mu, \phi^i) = \partial_\mu\phi^i(x^\mu) + \epsilon\partial_\mu\eta^i(x^\mu, \phi^i)....
\end{aligned} \tag{2.2.12}$$

In order to derive the general solution to the Noether variational problem, consider the variation given in equations (2.2.12) in the corresponding action we have

$$\delta S = \int_R \mathcal{L}(\tilde{x}^\mu, \tilde{\phi}^i, \partial_\mu\tilde{\phi}^i)d^4\tilde{x} - \int_R \mathcal{L}(x^\mu, \phi^i, \partial_\mu\phi^i)d^4x, \tag{2.2.13}$$

where the integration is taken over spacetimes region R. The second order tensors play a fundamental role in the mechanics of deformable bodies because deformation and internal forces characterizing the behaviour of deformable bodies are described mathematically by second order tensors such as strain and stress tensors. The second order tensors satisfy all the axioms of a vector space. We denote the vector space of all second order tensors by $\mathcal{L}$. Then equation (2.2.13) takes the form

$$\begin{aligned}
\delta S = \int_R [\mathcal{L}(x^\mu, \phi^i, \partial_\mu\phi^i) + \delta\mathcal{L}(x^\mu, \phi^i, \partial_\mu\phi^i)][1 + \partial_\mu(\delta x^\mu)]d^4x \\
- \int_R \mathcal{L}(x^\mu, \phi^i, \partial_\mu\phi^i)dx^4,
\end{aligned} \tag{2.2.14}$$

where the transformation of the volume element proceeds with respect to the Jacobian $\frac{\partial\tilde{x}}{\partial x}$ and is corrected to the first order in ϵ hence we have

$$\delta S = \int_{\Re} \left[\delta\mathcal{L}(x^\mu, \phi^i, \partial_\mu\phi^i) + \mathcal{L}(x^\mu, \phi^i, \partial_\mu\phi^i)\partial_\mu(\delta x^\mu)\right] d^4x. \tag{2.2.15}$$

Equation (2.2.15) can be written as

$$\delta S = \int_{\Re} \left[\frac{\partial\mathcal{L}}{\partial\phi^i}\delta_o\phi^i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\delta_o(\partial_\mu\phi^i) + (\partial_\mu\mathcal{L})\delta x^\mu + \mathcal{L}\partial_\mu(\delta x^\mu)\right] d^4x, \tag{2.2.16}$$

$$\delta S = \int_{\Re} \left[\frac{\partial\mathcal{L}}{\partial\phi^i}\delta_o\phi^i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\delta_o(\partial_\mu\phi^i) + \partial_\mu(\mathcal{L}\delta x^\mu)\right] d^4x. \tag{2.2.17}$$

We know that $\delta_o(\partial_\mu\phi^i) = \partial_\mu(\delta_o\phi^i)$, then equation (2.2.17) leads to

$$\delta S = \int_{\Re} \left[\frac{\partial\mathcal{L}}{\partial\phi^i}\delta_o\phi^i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\partial_\mu(\delta_o\phi^i) + \partial_\mu(\mathcal{L}\delta x^\mu)\right] d^4x, \tag{2.2.18}$$

$$\delta S = \int_{\Re} \left[\frac{\partial \mathcal{L}}{\partial \phi^i} \delta_o \phi^i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \delta_o \phi^i \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \right) \delta_o \phi^i + \partial_\mu (\mathcal{L} \delta x^\mu) \right] d^4x. \quad (2.2.19)$$

Rearranging the terms we have

$$\delta S = \int_{\Re} \left[\left(\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \right) \delta_o \phi^i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \delta_o \phi^i + \mathcal{L} \delta x^\mu \right) \right] d^4x, \quad (2.2.20)$$

which implies that if the first order variation in the action vanishes ($\delta S = 0$) for arbitrary region of integration, then we have the following solution to the Noether variational problem

$$\left(\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \right) \delta_o \phi^i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \delta_o \phi^i + \mathcal{L} \delta x^\mu \right) = 0. \quad (2.2.21)$$

This equation is a restriction on the Lagrangian that must be satisfied. If the Euler-Lagrange equations are satisfied then we have

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \delta_o \phi^i + \mathcal{L} \delta x^\mu \right) = 0. \quad (2.2.22)$$

Equations (2.2.21) and (2.2.22) are the crucial results from which Noether's theorem follows. These equations are often quoted as Noether's theorem. According to equation (2.2.22) the quantity

$$\Phi = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \delta_o \phi^i + \mathcal{L} \delta x^\mu, \quad (2.2.23)$$

is conserved. Now if instead of $\delta S = 0$, we take the divergence term

$$\delta S = \int \partial_\mu \delta A^\mu d^4x, \quad (2.2.24)$$

in equation (2.2.20) it would not change the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} = 0. \quad (2.2.25)$$

In such a situation equation (2.2.21) takes the form

$$\begin{aligned} \left(\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \right) \delta_o \phi^i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \delta_o \phi^i + \mathcal{L} \delta x^\mu \right) &= \partial_\mu \delta A^\mu, \\ \left(\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \right) \delta_o \phi^i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \delta_o \phi^i + \mathcal{L} \delta x^\mu - \delta A^\mu \right) &= 0. \end{aligned} \quad (2.2.26)$$

If the Euler-Lagrange equations are satisfied then the conserved quantity becomes

$$\Phi = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \delta_o \phi^i + \mathcal{L} \delta x^\mu - \delta A^\mu, \quad (2.2.27)$$

where A^{μ} is called the gauge function. Introducing

$$\delta_o \phi^i = \delta\phi^i - \delta x^{\nu}\partial_{\nu}\phi^i,$$

from equation (2.2.7) in equation (2.2.27), we have

$$\Phi = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi^i)}(\delta\phi^i - \delta x^{\nu}\partial_{\nu}\phi^i) + \mathcal{L}\delta x^{\mu} - \delta A^{\mu}. \tag{2.2.28}$$

Now using the transformation given by equations (2.2.12) the conserved quantity takes the usual form

$$\Phi = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi^i)}(\eta^i - \xi^{\nu}\partial_{\nu}\phi^i) + \mathcal{L}\xi^{\mu} - \delta A^{\mu}. \tag{2.2.29}$$

2.2.2 Noether Variational Problem in General Coordinates

For one independent variable t and m dependent variables q^i, where $i = 1, 2, ..., m$, the second order Lagrangian density takes the form $\mathcal{L}(t, q^i, \dot{q}^i, \ddot{q}^i)$, using this Lagrangian density then equation (2.2.26) takes the form [7, 69, 70]

$$\begin{aligned} &\left[\frac{\partial \mathcal{L}}{\partial q^i} - D\frac{\partial \mathcal{L}}{\partial \dot{q}^i} + D^2\frac{\partial \mathcal{L}}{\partial \ddot{q}^i}\right]\delta_o q^i + \\ &D\left[\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - D\frac{\partial \mathcal{L}}{\partial \ddot{q}^i}\right)\delta_o q^i + \frac{\partial \mathcal{L}}{\partial \ddot{q}^i}\delta_o \dot{q}^i + \mathcal{L}\delta t - \delta A\right] = 0. \end{aligned}$$

Where ($D = \frac{d}{dt}$). Let us see this equation for Lagrangian density of order three that is

$$\mathcal{L}(t, q^i, q^i_t, q^i_{tt}, q^i_{ttt}).$$

The above equation (2.2.26) takes the form

$$\begin{aligned} &\left[\frac{\partial \mathcal{L}}{\partial q^i} - D\frac{\partial \mathcal{L}}{\partial q^i_t} + D^2\frac{\partial \mathcal{L}}{\partial q^i_{tt}} - D^3\frac{\partial \mathcal{L}}{\partial q^i_{ttt}}\right]\delta_o q^i + \\ &D\left[\left(\frac{\partial \mathcal{L}}{\partial q^i_t} - D\frac{\partial \mathcal{L}}{\partial q^i_{tt}} + D^2\frac{\partial \mathcal{L}}{\partial q^i_{ttt}}\right)\delta_o q^i + \right. \\ &\left.\left(\frac{\partial \mathcal{L}}{\partial q^i_{tt}} - D\frac{\partial \mathcal{L}}{\partial q^i_{ttt}}\right)\delta_o q^i_t + \frac{\partial \mathcal{L}}{\partial q^i_{ttt}}\delta_o q^i_{tt} + \mathcal{L}\delta t - \delta A\right] = 0. \end{aligned} \tag{2.2.30}$$

Looking equation (2.2.30), we are in the position to write it for the Lagrangian density of order k

$$\mathcal{L}(t, q^i, q^i_t, q^i_{tt}, ..., q^i_{\underbrace{tt...t}_{k-times}}). \tag{2.2.31}$$

For Lagrangian density given in equation (2.2.31), equation (2.2.26) becomes

$$\left[\frac{\partial \mathcal{L}}{\partial q^i} - D\frac{\partial \mathcal{L}}{\partial q^i_t} + D^2\frac{\partial \mathcal{L}}{\partial q^i_{tt}} - D^3\frac{\partial \mathcal{L}}{\partial q^i_{ttt}} + ...(-1)^k D^k \frac{\partial \mathcal{L}}{\partial q^i_{\underbrace{tt...t}_{k-times}}}\right]\delta_o q^i +$$
$$D\Bigg[\left(\frac{\partial \mathcal{L}}{\partial q^i_t} - D\frac{\partial \mathcal{L}}{\partial q^i_{tt}} + D^2\frac{\partial \mathcal{L}}{\partial q^i_{ttt}} + ...(-1)^{k-1} D^{k-1} \frac{\partial \mathcal{L}}{\partial q^i_{\underbrace{tt...t}_{k-times}}}\right)\delta_o q^i +$$
$$\left(\frac{\partial \mathcal{L}}{\partial q^i_{tt}} - D\frac{\partial \mathcal{L}}{\partial q^i_{ttt}} + D^2\frac{\partial \mathcal{L}}{\partial q^i_{tttt}} + ...(-1)^{k-2} D^{k-2} \frac{\partial \mathcal{L}}{\partial q^i_{\underbrace{tt...t}_{k-times}}}\right)\delta_o q^i_t +$$
$$\left(\frac{\partial \mathcal{L}}{\partial q^i_{ttt}} - D\frac{\partial \mathcal{L}}{\partial q^i_{tttt}} + D^2\frac{\partial \mathcal{L}}{\partial q^i_{ttttt}} + ...(-1)^{k-3} D^{k-3} \frac{\partial \mathcal{L}}{\partial q^i_{\underbrace{tt...t}_{k-times}}}\right)\delta q^i_{tt} +$$
$$...... + \frac{\partial \mathcal{L}}{\partial q^i_{\underbrace{tt...t}_{k-times}}}\delta_o q^i_{\underbrace{tt...t}_{(k-1)-times}} + \mathcal{L}\delta t - \delta A\Bigg] = 0.$$

The Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial q^i} - D\frac{\partial \mathcal{L}}{\partial q^i_t} + D^2\frac{\partial \mathcal{L}}{\partial q^i_{tt}} - D^3\frac{\partial \mathcal{L}}{\partial q^i_{ttt}} + ...(-1)^k D^k \frac{\partial \mathcal{L}}{\partial q^i_{\underbrace{tt...t}_{k-times}}} = 0.$$

The corresponding conservation laws are

$$D\Bigg[\left(\frac{\partial \mathcal{L}}{\partial q^i_t} - D\frac{\partial \mathcal{L}}{\partial q^i_{tt}} + D^2\frac{\partial \mathcal{L}}{\partial q^i_{ttt}} + ...(-1)^{k-1} D^{k-1} \frac{\partial \mathcal{L}}{\partial q^i_{\underbrace{tt...t}_{k-times}}}\right)\delta_o q^i +$$
$$\left(\frac{\partial \mathcal{L}}{\partial q^i_{tt}} - D\frac{\partial \mathcal{L}}{\partial q^i_{ttt}} + D^2\frac{\partial \mathcal{L}}{\partial q^i_{tttt}} + ...(-1)^{k-2} D^{k-2} \frac{\partial \mathcal{L}}{\partial (q^i_{\underbrace{tt...t}_{k-times}})}\right)\delta_o q^i_t +$$
$$\left(\frac{\partial \mathcal{L}}{\partial q^i_{ttt}} - D\frac{\partial \mathcal{L}}{\partial q^i_{tttt}} + D^2\frac{\partial \mathcal{L}}{\partial q^i_{ttttt}} + ...(-1)^{k-3} D^{k-3} \frac{\partial \mathcal{L}}{\partial q^i_{\underbrace{tt...t}_{k-times}}}\right)\delta_o q^i_2 +$$
$$...... + \frac{\partial \mathcal{L}}{\partial q^i_{\underbrace{tt...t}_{k-times}}}\delta_o q^i_{\underbrace{tt...t}_{(k-1)-times}} + \mathcal{L}\delta t - \delta A\Bigg] = 0.$$

Now we want to write these equations for n independent and m dependent variables and Lagrangian of order k [10] that is

$$\begin{aligned} &x = (x^1, x^2, ..., x^n), \quad q = (q^1, q^2, ..., q^m), \\ &\mathcal{L}(x, q, q_{i_1}, q_{i_1 i_2}, ..., q_{i_1 i_2 ... i_k}). \end{aligned} \tag{2.2.32}$$

For Lagrangian given in equation (2.2.32), the equation (2.2.26) takes the form

$$\left[\frac{\partial\mathcal{L}}{\partial q^i}-D_{i_1}\frac{\partial\mathcal{L}}{\partial q^i_{i_1}}+D_{i_1}D_{i_2}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2}}-D_{i_1}D_{i_2}D_{i_3}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3}}+..(-1)^kD_{i_1}..D_{i_k}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2..i_k}}\right]\delta_oq^i+$$
$$D_{i_1}\left[\left(\frac{\partial\mathcal{L}}{\partial q^i_{i_1}}-D_{i_2}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2}}+D_{i_2}D_{i_3}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3}}+...(-1)^{k-1}D_{i_2}D_{i_3}...D_{i_k}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2..i_k}}\right)\delta_oq^i+\right.$$
$$\left(\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2}}-D_{i_3}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3}}+D_{i_3}D_{i_4}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3i_4}}+...(-1)^{k-2}D_{i_3}D_{i_4}...D_{i_k}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3..i_k}}\right)\delta_oq^i_{i_2}+$$
$$\left(\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3}}-D_{i_4}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3i_4}}+D_{i_4}D_{i_5}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3i_4i_5}}+...(-1)^{k-3}D_{i_4}D_{i_5}...D_{i_k}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3..i_k}}\right)\delta_oq^i_{i_2i_3}$$
$$\left.+......+\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3..i_k}}\delta_oq^i_{i_2i_3i_4..i_k}+\mathcal{L}\delta x^{i_1}-\delta A^{i_1}\right]=0.$$

The corresponding Euler-Lagrange equations are

$$\frac{\partial\mathcal{L}}{\partial q^i}-D_{i_1}\frac{\partial\mathcal{L}}{\partial q^i_{i_1}}+D_{i_1}D_{i_2}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2}}-D_{i_1}D_{i_2}D_{i_3}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3}}+..(-1)^kD_{i_1}..D_{i_k}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2..i_k}}=0. \quad (2.2.33)$$

And the conservation laws take the form

$$D_{i_1}\left[\left(\frac{\partial\mathcal{L}}{\partial q^i_{i_1}}-D_{i_2}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2}}+D_{i_2}D_{i_3}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3}}+...(-1)^{k-1}D_{i_2}D_{i_3}...D_{i_k}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2..i_k}}\right)\delta_oq^i+\right.$$
$$\left(\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2}}-D_{i_3}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3}}+D_{i_3}D_{i_4}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3i_4}}+...(-1)^{k-2}D_{i_3}D_{i_4}...D_{i_k}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3..i_k}}\right)\delta_oq^i_{i_2}+$$
$$\left(\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3}}-D_{i_4}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3i_4}}+D_{i_4}D_{i_5}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3i_4i_5}}+...(-1)^{k-3}D_{i_4}D_{i_5}...D_{i_k}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3..i_k}}\right)\delta_oq^i_{i_2i_3}$$
$$\left.+......+\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3..i_k}}\delta_oq^i_{i_2i_3i_4..i_k}+\mathcal{L}\delta x^{i_1}-\delta A^{i_1}\right]=0.$$

If the action

$$S=\int_R\mathcal{L}(x,q,q_{i_1},q_{i_1i_2},...,q_{i_1i_2...i_k})dx, \quad (2.2.34)$$

is invariant under the symmetry transformations

$$\begin{aligned}
x^* &\to x+\epsilon\xi(x,q,q_{i_1},q_{i_1i_2}...q_{i_1i_2...i_p})+...,\\
q^* &\to q+\epsilon\eta(x,q,q_{i_1},q_{i_1i_2}...q_{i_1i_2...i_p})+...,\\
q^*_{i_1} &\to q_{i_1}+\epsilon\eta_{i_1}(x,q,q_{i_1},q_{i_1i_2}...q_{i_1i_2...i_p})+...,\\
&\vdots\\
q^*_{i_1i_2...i_k} &\to q_{i_1i_2...i_k}+\epsilon\eta_{i_1i_2...i_k}(x,q,q_{i_1},q_{i_1i_2}...q_{i_1i_2...i_p})+....
\end{aligned} \quad (2.2.35)$$

then the quantity

$$\left[\frac{\partial\mathcal{L}}{\partial q^i_{i_1}} - D_{i_2}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2}} + D_{i_2}D_{i_3}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3}} + ...(-1)^{k-1}D_{i_2}D_{i_3}...D_{i_k}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2..i_k}}\right]\delta_o q^i +$$
$$\left(\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2}} - D_{i_3}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3}} + D_{i_3}D_{i_4}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3i_4}} + ...(-1)^{k-2}D_{i_3}D_{i_4}...D_{i_k}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3..i_k}}\right)\delta_o q^i_{i_2} +$$
$$\left(\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3}} - D_{i_4}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3i_4}} + D_{i_4}D_{i_5}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3i_4i_5}} + ...(-1)^{k-3}D_{i_4}D_{i_5}...D_{i_k}\frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3..i_k}}\right)\delta_o q^i_{i_2i_3}$$
$$+ ...... + \frac{\partial\mathcal{L}}{\partial q^i_{i_1i_2i_3..i_k}}\delta_o q^i_{i_2i_3i_4..i_k} + \mathcal{L}\delta x^{i_1} - \delta A^{i_1},$$

is conserved.

2.2.3 Noether Symmetry Equation

For action defined in equation (2.2.34) and transformations given by equations (2.2.35), equation (2.2.26) takes the form

$$\frac{\partial\mathcal{L}}{\partial x}\delta x + \frac{\partial\mathcal{L}}{\partial q}\delta_o q + \frac{\partial\mathcal{L}}{\partial q_{i_1}}\delta_o q_{i_1} + ... + \frac{\partial\mathcal{L}}{\partial q_{i_1i_2...i_k}}\delta_o q_{i_1i_2...i_k} + \mathcal{L}D\delta x - DA = 0. \tag{2.2.36}$$

Now using the notation of transformations given by equations (2.2.35), we have

$$\xi\frac{\partial\mathcal{L}}{\partial x} + \eta\frac{\partial\mathcal{L}}{\partial q} + \eta_{i_1}\frac{\partial\mathcal{L}}{\partial q_{i_1}} + ... + \eta_{i_1i_2...i_k}\frac{\partial\mathcal{L}}{\partial q_{i_1i_2...i_k}} + \mathcal{L}D\xi - DA = 0 > \tag{2.2.37}$$

This equation can be written as

$$\left[\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial q} + \eta_{i_1}\frac{\partial}{\partial q_{i_1}} + ... + \eta_{i_1i_2...i_k}\frac{\partial}{\partial q_{i_1,i_2...i_k}}\right]\mathcal{L} + \mathcal{L}D\xi = DA. \tag{2.2.38}$$

This is the most general form of Noether symmetry equation or Noether symmetry condition. If we define

$$\mathbf{X} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial q}, \tag{2.2.39}$$

and its k^{th} order prolongation

$$\mathbf{X}^{[k]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial q} + \eta_{i_1}\frac{\partial}{\partial q_{i_1}} + ... + \eta_{i_1i_2...i_k}\frac{\partial}{\partial q_{i_1,i_2...i_k}}, \tag{2.2.40}$$

then equation (2.2.38) takes the form

$$\mathbf{X}^{[k]}\mathcal{L} + \mathcal{L}D\xi = DA. \tag{2.2.41}$$

$\mathbf{X}$ defined in equation (2.2.39) is called Noether symmetry generator, and the equation (2.2.41) is compact form of Noether symmetry equation. The operator D is the total derivative operator defined by the equation

$$D = \frac{\partial}{\partial x} + q_{i_1}\frac{\partial}{\partial q} + q_{i_1 i_2}\frac{\partial}{\partial q_{i_1}} + ... + q_{i_1 i_2 ... i_k}\frac{\partial}{\partial q_{i_1, i_2 ... i_{k-1}}}.$$

2.2.4 Noether's Theorem

If $\mathbf{X}$ defined in equation (2.2.39) generates the variational symmetry for action defined in equation (2.2.34) then the quantity [45, 50]

$$\begin{aligned} I = \delta A^{i_1} - &\left[\frac{\partial \mathcal{L}}{\partial q_i^{i_1}} - D_{i_2}\frac{\partial \mathcal{L}}{\partial q^i_{i_1 i_2}} + D_{i_2}D_{i_3}\frac{\partial \mathcal{L}}{\partial q^i_{i_1 i_2 i_3}} + ...(-1)^{k-1}D_{i_2}D_{i_3}...D_{i_k}\frac{\partial \mathcal{L}}{\partial q^i_{i_1 i_2 .. i_k}}\right]\delta_o q^i - \\ &\left(\frac{\partial \mathcal{L}}{\partial q^i_{i_1 i_2}} - D_{i_3}\frac{\partial \mathcal{L}}{\partial q^i_{i_1 i_2 i_3}} + D_{i_3}D_{i_4}\frac{\partial \mathcal{L}}{\partial q^i_{i_1 i_2 i_3 i_4}} + ...(-1)^{k-2}D_{i_3}D_{i_4}...D_{i_k}\frac{\partial \mathcal{L}}{\partial q^i_{i_1 i_2 i_3 .. i_k}}\right)\delta_o q^i_{i_2} \\ &- \left(\frac{\partial \mathcal{L}}{\partial q^i_{i_1 i_2 i_3}} - D_{i_4}\frac{\partial \mathcal{L}}{\partial q^i_{i_1 i_2 i_3 i_4}} + D_{i_4}D_{i_5}\frac{\partial \mathcal{L}}{\partial q^i_{i_1 i_2 i_3 i_4 i_5}} + ...(-1)^{k-3}D_{i_4}D_{i_5}...D_{i_k}\frac{\partial \mathcal{L}}{\partial q^i_{i_1 i_2 i_3 .. i_k}}\right)\delta_o q^i_{i_2 i_3} \\ &- ...... - \frac{\partial \mathcal{L}}{\partial q^i_{i_1 i_2 i_3 .. i_k}}\delta_o q^i_{i_2 i_3 i_4 .. i_k} - \mathcal{L}\delta x^{i_1}, \end{aligned} \tag{2.2.42}$$

is conserved corresponding to the Euler-Lagrange equations [68]

$$\frac{\partial \mathcal{L}}{\partial q^i} - D_{i_1}\frac{\partial \mathcal{L}}{\partial q^i_{i_1}} + D_{i_1}D_{i_2}\frac{\partial \mathcal{L}}{\partial q^i_{i_1 i_2}} - D_{i_1}D_{i_2}D_{i_3}\frac{\partial \mathcal{L}}{\partial q^i_{i_1 i_2 i_3}} + ..(-1)^k D_{i_1}..D_{i_k}\frac{\partial \mathcal{L}}{\partial q^i_{i_1 i_2 .. i_k}} = 0. \tag{2.2.43}$$

2.3 Approximate Noether Symmetry

2.3.1 First Order Approximation

The first order approximate Noether symmetry generator for $\mathbf{x} = (x^1, x^2, ..., x^n)$ and $\mathbf{u} = (u^1, u^2, ..., u^m)$ is defined as [42]

$$\mathbf{X} = \mathbf{X}_0 + \epsilon \mathbf{X_1}. \tag{2.3.1}$$

Explicitly we can write

$$\mathbf{X} = (\xi_0^i + \epsilon \xi_1^i)\frac{\partial}{\partial x^i} + (\eta_0^j + \epsilon \eta_1^j)\frac{\partial}{\partial u^j}, \tag{2.3.2}$$

and its first order prolongation is

$$\mathbf{X}^{[1]} = (\xi_0^i + \epsilon \xi_1^i)\frac{\partial}{\partial x^i} + (\eta_0^j + \epsilon \eta_1^j)\frac{\partial}{\partial u^j} + (\eta_0^j + \epsilon \eta_1^j)_{i_1}\frac{\partial}{\partial u^j_{i_1}}. \tag{2.3.3}$$

The Lagrangian density perturbed up to the first order in ϵ is

$$\mathcal{L} = \mathcal{L}_0(x, u, u_{i_1}) + \epsilon\mathcal{L}_1(x, u, u_{i_1}). \tag{2.3.4}$$

Then the approximate Noether symmetry equation reads

$$\mathbf{X}^{[1]}\mathcal{L} + (D_i\xi^i)\mathcal{L} = D_iA^i, \tag{2.3.5}$$

which splits into two equations

$$\begin{aligned}
&\mathbf{X_0}^{[1]}\mathcal{L} + (D_i\xi^i)\mathcal{L} = D_iA_0^i,\\
&\mathbf{X_0^{[1]}}\mathcal{L}_1 + \mathbf{X_1^{[1]}}\mathcal{L}_0 + (D_i\xi_0^i)\mathcal{L}_1 + (D_i\xi_1^i)\mathcal{L}_0 = D_iA_1^i.
\end{aligned} \tag{2.3.6}$$

The second part of equations (2.3.6) gives the first order approximate Noether symmetry corresponding to the given Lagrangian.

2.3.2 K^{th} Order Approximate Noether Symmetry

The approximate Noether symmetry generator of order k in ϵ for $\mathbf{x} = (x^1, x^2, ..., x^n)$ and $\mathbf{u} = (u^1, u^2, ..., u^m)$ is defined as [42]

$$\mathbf{X} = \mathbf{X}_0 + \epsilon\mathbf{X_1} + \epsilon^2\mathbf{X_2} + ... + +\epsilon^k\mathbf{X_k}. \tag{2.3.7}$$

Explicitly we can write

$$\mathbf{X} = (\xi_0^i + \epsilon\xi_1^i + ... + \epsilon^k\xi_k^i)\frac{\partial}{\partial x_i} + (\eta_0^j + \epsilon\eta_1^j + ... + \epsilon^k\eta_k^j)\frac{\partial}{\partial u^j}, \tag{2.3.8}$$

and its p^{th} order prolongation is

$$\begin{aligned}
\mathbf{X}^{[p]} &= (\xi_0^i + \epsilon\xi_1^i + ... + \epsilon^k\xi_k^i)\frac{\partial}{\partial x^i} + (\eta_0^j + \epsilon\eta_1^j + ... + \epsilon^k\eta_k^j)\frac{\partial}{\partial u^j}\\
&+ (\eta_j^0 + \epsilon\eta_1^j + ... + \epsilon^k\eta_k^j)_{i_1}\frac{\partial}{\partial u_{i_1}^j} + ... + (\eta_0^j + \epsilon\eta_1^j + ... + \epsilon^k\eta_k^j)_{i_1i_2...i_p}\frac{\partial}{\partial u_{i_1i_2...i_p}^j}.
\end{aligned} \tag{2.3.9}$$

The perturbed Lagrangian of order k in ϵ is

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0(x, u, u_{i_1}, ..., u_{i_1i_2...i_p}) + \epsilon\mathcal{L}_1(x, u, u_{i_1}, ..., u_{i_1,i_2...i_p})\\
&+ \epsilon^2\mathcal{L}_2(x, u, u_{i_1}, ..., u_{i_1i_2...i_p}) + ... + \epsilon^k\mathcal{L}_k(x, u, u_{i_1}, ..., u_{i_1i_2...i_p}).
\end{aligned} \tag{2.3.10}$$

Then the approximate Noether symmetry equation reads

$$\mathbf{X}^{[p]}L + (D_i\xi^i)L = D_iA^i, \tag{2.3.11}$$

where

$$A^i = A_0^i + \epsilon A_1^i + \epsilon^2 A_2^i + ... + \epsilon^k A_k^i. \tag{2.3.12}$$

Equation (2.3.11) splits into $(k+1)$ equations.

Example 9: The Lagrangian density corresponding to differential equation (1.6.10) is

$$\mathcal{L} = \frac{y'^2}{2}, \tag{2.3.13}$$

and the first order prolonged symmetry generator for this Lagrangian density is

$$\mathbf{X}^{[1]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta'(x,y,y')\frac{\partial}{\partial y'}. \tag{2.3.14}$$

Using equations (2.3.13) and (2.3.14) in equation (2.3.5), we have the following system of determining PDEs,

$$\xi_y(x,y) = 0, \quad A_x(x,y) = 0, \quad \eta_x(x,y) - A_y = 0, \quad 2\eta_y(x,y) - \xi_x(x,y) = 0. \tag{2.3.15}$$

The solution of this system is

$$\begin{aligned} &A(x,y) = \frac{c_1 y^2}{2} + c_2 y + c_3, \quad \xi(x,y) = c_1 x^2 + c_4 x + c_5, \\ &\eta(x,y) = c_1 xy + \frac{c_4 y}{2} + c_2 x + c_6. \end{aligned}$$

The Noether symmetry generators, gauge functions and first integrals are given below

No.	N. Symmetry	Gauge Functions	First Integral
1.	$\mathbf{X_1} = \frac{\partial}{\partial x}$	$A_1 = 0$	$\phi_1 = \frac{y'^2}{2}$
2.	$\mathbf{X_2} = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$A_2 = 0$	$\phi_2 = y'(xy' - y)$
3.	$\mathbf{X_3} = \frac{\partial}{\partial y}$	$A_3 = 0$	$\phi_3 = y'$
4.	$\mathbf{x_4} = x\frac{\partial}{\partial y}$	$A_4 = y$	$\phi_4 = y - xy'$
5.	$\mathbf{X_5} = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$	$A_5 = \frac{y^2}{2}$	$\phi_5 = \frac{y^2}{2} + \frac{x^2 y'^2}{2} - xyy'$

Now consider the first order perturbed Lagrangian density [55]

$$\mathcal{L} = \frac{y'^2 - \epsilon y^2}{2}, \tag{2.3.16}$$

corresponding to differential equation (1.6.11) and the approximate symmetry generator

$$\mathbf{X}^{[1]} = (\xi_0 + \epsilon\xi_1)\frac{\partial}{\partial x} + (\eta_0 + \epsilon\eta_1)\frac{\partial}{\partial y} + (\eta_0' + \epsilon\eta_1')\frac{\partial}{\partial y'}. \tag{2.3.17}$$

Using equations (2.3.16) and (2.3.17) in equation (2.3.5) and collecting the system of PDEs up to first order in ϵ, then solving it we get Noether symmetries with approximate parts, along with approximate gauge functions, and approximate first integrals, which are given in the following table

No.	N. Symmetry	Gauge Functions	First Integral
1.	$\mathbf{X_1} = \frac{\partial}{\partial x}$	$A_1 = 0$	$\phi_1 = \frac{y'^2+\epsilon y^2}{2}$
2.	$\mathbf{X_2} = (2x - \epsilon\frac{4x^3}{3})\frac{\partial}{\partial x} + (y - \epsilon 2x^2)\frac{\partial}{\partial y}$	$A_2 = -2\epsilon xy^2$	$\phi_2 = y'[xy' - y] - \epsilon x(y^2 - 2xyy' + \frac{2x^2y^2}{3})$
3.	$\mathbf{X_3} = (1 - \epsilon\frac{x^2}{2})\frac{\partial}{\partial y}$	$A_3 = -\epsilon xy$	$\phi_3 = (1 - \epsilon\frac{x^2}{2})y' + \epsilon xy$
4.	$\mathbf{X_4} = (x - \epsilon\frac{x^3}{6})\frac{\partial}{\partial y}$	$A_4 = (1 - \epsilon\frac{x^2}{2})y$	$\phi_4 = (1 - \epsilon\frac{x^2}{2})y - (x - \epsilon\frac{x^3}{6})y'$
5.	$\mathbf{X_5} = (2x^2 - \epsilon\frac{2x^4}{3})\frac{\partial}{\partial x} + y(2x - \epsilon\frac{4x^3}{3})\frac{\partial}{\partial y}$	$A_5 = (1 - \epsilon 2x^2)y^2$	$\phi_5 = (y - xy')^2 - \epsilon\frac{x^2}{3}(3y - xy')(y - xy')$
6.	$\mathbf{X_6} = \epsilon\frac{\partial}{\partial x}$	$A_6 = 0$	$\phi_6 = \epsilon\frac{y'^2}{2}$
7.	$\mathbf{X_7} = \epsilon(2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$	$A_7 = 0$	$\phi_7 = \epsilon y'(xy' - y)$
8.	$\mathbf{X_8} = \epsilon\frac{\partial}{\partial y}$	$A_8 = 0$	$\phi_8 = \epsilon y'$
9.	$\mathbf{X_9} = \epsilon x\frac{\partial}{\partial y}$	$A_9 = \epsilon y$	$\phi_9 = \epsilon(y - xy')$
10.	$\mathbf{X_{10}} = \epsilon(2x\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y})$	$A = \epsilon y^2$	$\phi_{10} = \epsilon(y - xy')^2$

Chapter 3

Noether Symmetries of the Arc Length Minimizing Lagrangian of Plane Symmetric Static Spacetimes

3.1 Introduction

In this chapter we find symmetries of the arc length minimizing Lagrangian of plane symmetric static spacetimes [12–15, 32, 35, 61, 62, 72, 73]. These are the spacetimes admitting $[SO(2)\bigotimes_s \mathbb{R}^2]\bigotimes \mathbb{R}$ as the minimal isometry group in such a way that the group orbits are two dimensional hypersurfaces of zero curvature. Here $SO(2)$ corresponds to rotation and $\mathbb{R}^2$ to translations along spatial directions y and z, $\bigotimes_s$ and $\bigotimes$ stand for the semidirect product and the direct product respectively. Noether proved [17, 58] that for every Noether symmetry there is a conservation law (conserved quantity). For example, symmetries under spatial translations imply conservation of linear momentums, symmetry under time translation implies conservation of energy, and symmetry under rotation implies conservation of angular momentum.

The solution of EFEs in closed analytic form is impossible. Different methods are used to find the exact solutions of EFEs [66]. We use Noether symmetries to classify the spacetimes and find all plane symmetric static solutions of EFEs [21]. Using the Noether's theorem the first integrals corresponding to Noether symmetries are also calculated in this chapter and chapters 4, 5 and 6.

For the general plane symmetric static spacetime [13]

$$ds^2 = e^{\nu(x)}dt^2 - dx^2 - e^{\mu(x)}(dy^2 + dz^2), \tag{3.1.1}$$

the corresponding arc length minimizing Lagrangian density takes the form [25]

$$\mathcal{L} = e^{\nu(x)}\dot{t}^2 - \dot{x}^2 - e^{\mu(x)}(\dot{y}^2 + \dot{z}^2), \tag{3.1.2}$$

where " $\cdot$ " denotes differentiation with respect to s and the metric coefficients $\nu(x)$ and $\mu(x)$ are arbitrary functions of x. We intend to find all possible values for these arbitrary functions $\nu(x)$ and $\mu(x)$. For this purpose we use Noether symmetry condition given by equation (2.2.41) and find the Noether symmetries of the action corresponding to the Lagrangian given by equation (3.1.2). Other than the cases that give minimal set of Noether symmetries, different $\nu(x)$ and $\mu(x)$ correspond to a different sets of Noether symmetries. Hence for each set of $\nu(x)$ and $\mu(x)$ we have a unique set of Noether symmetries. Therefore, from this procedure we get classification of the geodesic Lagrangians of the plane symmetric static spacetimes. Once we know the geodesic Lagrangian, we can write the corresponding spacetime easily. This classification recovers the existing cases of plane symmetric static spacetime [21, 32, 51] along with first integrals corresponding to each Noether symmetry.

3.2 The Noether Symmetry Governing Equations

The first order prolonged generator $\mathbf{X}^{[\mathbf{1}]}$ in the Noether symmetry equation (2.2.41) takes the form

$$\mathbf{X}^{[\mathbf{1}]} = \mathbf{X} + \eta^i_s \frac{\partial}{\partial \dot{x}^i}, \tag{3.2.1}$$

where x^i refers to the dependent variables (t, x, y, z) and

$$\mathbf{X} = \xi\frac{\partial}{\partial t} + \eta^i \frac{\partial}{\partial x^i}, \tag{3.2.2}$$

where, ξ and η^i are functions of s, t, x, y, z and η^i_s are functions of $s, t, x, y, z, \dot{t}, \dot{x}, \dot{y}.\dot{z}$ [3, 16, 38, 63] and determined by

$$\eta^i_s = D(\eta^i) - \dot{x}^i D(\xi). \tag{3.2.3}$$

The differential operator D in equation (2.2.41) becomes

$$D = \frac{\partial}{\partial s} + \dot{x}^i \frac{\partial}{\partial x^i}. \tag{3.2.4}$$

The conservation law or the first integral given in equation (2.2.42), for the first order Lagrangian given by equation (3.1.2) takes the form

$$I = A - \frac{\partial L}{\partial(\dot{x}^i)}(\eta^i - \xi\dot{x}^i) - L\xi. \tag{3.2.5}$$

The expression given in equation (3.2.5) will be used to find first integrals corresponding to each Noether symmetry.

3.3 Determining PDEs System

Using the first order prolonged symmetry generator given by equation (3.2.1), Lagrangian given by equation (3.1.2), and differential operator given by equation (3.2.4), in equation (2.2.41), we get the following system of 19 PDEs

$$\begin{aligned}
&\xi_t = 0, \quad \xi_x = 0, \quad \xi_y = 0, \quad \xi_z = 0, \quad A_s = 0,\\
&2e^{\nu(x)}\eta^0_s = A_t, \quad -2\eta^1_s = A_x,\\
&-2e^{\mu(x)}\eta^2_s = A_y, \quad -2e^{\mu(x)}\eta^3_s = A_z,\\
&2\eta^1_x - \xi_s = 0, \quad \eta^2_z + \eta^3_y = 0,\\
&\eta^1_y + e^{\mu(x)}\eta^2_x = 0, \quad \eta^1_z + e^{\mu(x)}\eta^3_x = 0,\\
&-\eta^1_t + e^{\nu(x)}\eta^0_x = 0, \quad e^{\nu(x)}\eta^0_y - e^{\mu(x)}\eta^2_t = 0,\\
&e^{\nu(x)}\eta^0_z - e^{\mu(x)}\eta^3_t = 0, \quad \nu^{'}(x)\eta^1 + 2\eta^0_t - \xi_s = 0,\\
&\mu^{'}(x)\eta^1 + 2\eta^2_y - \xi_s = 0, \quad \mu^{'}(x)\eta^1 + 2\eta^3_z - \xi_s = 0,
\end{aligned} \tag{3.3.1}$$

where " $'$ " is the derivative with respect to x.

In this system, $\nu(x)$, $\mu(x)$, $\xi(s,t,x,y,z)$, $A(s,t,x,y,z)$, and $\eta^i(s,t,x,y,z)$, $i = 0,1,2,3$ are unknown functions. $\nu(x)$ and $\mu(x)$ are the metric coefficients of the plane symmetric static spacetimes, $\xi(s,t,x,y,z)$, $\eta^i(s,t,x,y,z)$, $i = 0,1,2,3$ are components of Noether symmetry generators and $A(s,t,x,y,z)$ is the gauge function.

Lists of solutions of the above system are given in different sections with respect to different number of Noether symmetries along with the corresponding spacetimes. In each case the first integral corresponding to each Noether symmetry is also given. For the distinction between isometries and other Noether symmetries we use different notation, that is $\mathbf{X}_i$ for isometries and $\mathbf{Y}_j$ for non-isometries.

3.4 Five Noether Symmetries and their First Integrals

It is well known that the group of Killng vectors is a subgroup of the group of Noether symmetries. For plane symmetric static spacetimes the minimal set of Killing vectors contains four elements. Further, as $\frac{\partial}{\partial s}$ satisfies the Noether symmetry equation (2.2.41), it is also a Noether symmetry generator. Therefore, the minimal set of Noether symmetries for action of the arc length minimizing Lagrangian given by equation (3.1.2) for plane symmetric static spacetime consists of 5 Noether symmetry generators, and they are

$$\mathbf{X_0} = \frac{\partial}{\partial t}, \quad \mathbf{X_1} = \frac{\partial}{\partial y}, \quad \mathbf{X_2} = \frac{\partial}{\partial z}, \quad \mathbf{X_3} = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad \mathbf{Y_0} = \frac{\partial}{\partial s}, \tag{3.4.1}$$

where $\mathbf{X_0}$, $\mathbf{X_1}$, $\mathbf{X_2}$ and $\mathbf{X_3}$ are Killing vector fields admitted by the spacetime given in equation (3.1.1) and $\mathbf{Y_0}$ is the Noether symmetry which leaves the arc length minimizing Lagrangian invariant. There are infinitely many spacetimes admitting minimal set of Killing vectors. Therefore, there are infinitely many number of Lagrangian of plane symmetric static spacetime admitting minimal set of Noether symmetries, i.e. five Noether symmetries. All of them cannot be listed, however some of them, which appear in our calculation are given in Table 3.2. The first integrals corresponding to the set of Noether symmetries given by equations (3.4.1) are given in Table 3.1.

Table 3.1: First Integrals for the minimal set

Gen	First Integrals
$\mathbf{X_0}$	$\phi_0 = -2e^{\nu(x)}\dot{t}$
$\mathbf{X_1}$	$\phi_1 = 2e^{\mu(x)}\dot{y}$
$\mathbf{X_2}$	$\phi_2 = 2e^{\mu(x)}\dot{z}$
$\mathbf{X_3}$	$\phi_3 = 2e^{\mu(x)}(y\dot{z} - z\dot{y})$
$\mathbf{Y_0}$	$\phi_4 = e^{\nu(x)}\dot{t}^2 - \dot{x}^2 - e^{\mu(x)}(\dot{y}^2 + \dot{z}^2) = \mathcal{L}$

The Lie algebra of the Noether symmetry generators given in equations (3.4.1) is $[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_i}, \mathbf{X_j}] = 0$ and $[\mathbf{X_i}, \mathbf{Y_0}] = 0$ otherwise.

Table 3.2: Metric coefficients for five symmetries

No.	$\nu(x)$	$\mu(x)$
1.	$2\ln(\frac{x}{\alpha})$	$\frac{x}{\alpha}$
2.	$\frac{x}{\alpha}$	$2\ln(\frac{x}{\alpha})$
3.	$2\ln(\frac{x}{\alpha})$	$2\ln\cosh(\frac{x}{\alpha})$
4.	$2\ln(\frac{x}{\alpha})$	$2\ln\cos(\frac{x}{\alpha})$
5.	$2\ln\cosh(\frac{x}{\alpha})$	$2\ln(\frac{x}{\alpha})$
6.	$2\ln\cos(\frac{x}{\alpha})$	$2\ln(\frac{x}{\alpha})$
7.	$2\ln\cosh(\frac{x}{\alpha})$	$\frac{x}{\alpha}$
8.	$2\ln\cos(\frac{x}{\alpha})$	$\frac{x}{\alpha}$
9.	$\nu^{'}(x) \neq 0, \nu(x) \neq \mu(x)$	$2\ln\cosh(\frac{x}{\alpha}), 2\ln\cos(\frac{x}{\alpha})$
10.	$2\ln\cosh(\frac{x}{\alpha}), 2\ln\cos(\frac{x}{\alpha})$	$\nu(x) \neq \mu(x), \mu^{'}(x) \neq 0$
11.	$\nu^{''}(x) \neq 0$	$\frac{x}{\alpha}$
12.	$2\ln\frac{x}{\alpha}$	$\mu^{''}(x) \neq 0, \mu(x) \neq a\ln\frac{x}{\alpha}$
13.	$\nu(x) \neq \mu(x), \nu^{'}(x) \neq 0$	$\mu^{''}(x) \neq 0, \mu(x) \neq a\ln\frac{x}{\alpha}$
14.	$\nu^{''}(x) \neq 0, \nu(x) \neq a\ln\frac{x}{\alpha}$	$\nu(x) \neq \mu(x), \mu^{'}(x) \neq 0$

3.5 Six Noether Symmetries and First Integrals

Solutions of the system of equations (3.3.1) for action with geodesic Lagrangian given by equation (3.1.2) admitting six Noether symmetries are given bellow in this section:

Solution-I:

The metric coefficients are

$$\nu(x) = \frac{x}{\alpha}, \quad \mu(x) = \frac{x}{\beta}, \quad \alpha \neq \beta.$$

Components of the Noether symmetry generators are

$$\xi = c_1, \quad \eta^0 = -c_4\frac{t}{2\alpha} + c_2, \quad \eta^1 = c_4, \quad \eta^2 = -c_4\frac{y}{2\beta} - c_3 z + c_7,$$
$$\eta^3 = -c_4\frac{z}{2\beta} + c_3 y + c_5, \quad A = c_6.$$

The corresponding metric is

$$ds^2 = e^{\frac{x}{\alpha}}dt^2 - dx^2 - e^{\frac{x}{\beta}}(dy^2 + dz^2), \quad \alpha, \beta \neq 0. \tag{3.5.1}$$

The additional Noether symmetry to the set given by equations (3.4.1) is

$$\mathbf{X_4} = \frac{\partial}{\partial x} - \frac{t}{2\alpha}\frac{\partial}{\partial t} - \frac{y}{2\beta}\frac{\partial}{\partial y} - \frac{z}{2\beta}\frac{\partial}{\partial z}.$$

It is an isometry of metric given by equation (3.5.1). The corresponding first integral is

$$\phi_5 = \frac{t\dot{t}}{\alpha}e^{\frac{x}{\alpha}} + 2\dot{x} - \frac{e^{\frac{x}{\alpha}}}{\beta}(y\dot{y} + z\dot{z}).$$

The Lie algebra of $\mathbf{X_4}$ along with the symmetries given in equations (3.4.1) is
$[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_0}, \mathbf{X_4}] = -\frac{1}{2\alpha}\mathbf{X_0}$, $[\mathbf{X_1}, \mathbf{X_4}] = -\frac{1}{2\beta}\mathbf{X_1}$, $[\mathbf{X_2}, \mathbf{X_4}] = -\frac{1}{2\beta}\mathbf{X_2}$, $[\mathbf{X_i}, \mathbf{X_j}] = 0$ and $[\mathbf{X_i}, \mathbf{Y_0}] = 0$ otherwise.

Solution-II:

Coefficients of the metric are

$$\nu(x) = c, \quad \mu(x) = 2\ln\cosh\left(\frac{x}{\alpha}\right), \quad or \quad \mu(x) = 2\ln\cos\left(\frac{x}{\alpha}\right).$$

Components of the Noether symmetry generators are

$$\xi = c_1, \quad \eta^0 = c_2 s + c_3, \quad \eta^1 = 0, \quad \eta^2 = -c_4 z + c_5,$$
$$\eta^3 = c_4 y + c_6, \quad A = 2c_2 t + c_7.$$

The spacetimes in this case are

$$ds^2 = dt^2 - dx^2 - \cosh^2\left(\frac{x}{\alpha}\right)(dy^2 + dz^2), \quad \alpha \neq 0, \tag{3.5.2}$$

$$ds^2 = dt^2 - dx^2 - \cos^2\left(\frac{x}{\alpha}\right)(dy^2 + dz^2), \quad \alpha \neq 0. \tag{3.5.3}$$

The symmetry other than the minimal set is

$$\mathbf{Y_1} = s\frac{\partial}{\partial t}, \quad A = 2t. \tag{3.5.4}$$

It is neither Killing vector nor homothetic vector. First integral corresponding to $\mathbf{Y_1}$ is

$$\phi_5 = 2(t - s\dot{t}).$$

The Lie algebra of $\mathbf{Y_1}$ along with symmetries of equation (3.4.1) is

$[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{Y_0}, \mathbf{Y_1}] = \mathbf{X_0}$, $[\mathbf{X_i}, \mathbf{X_j}] = 0$ and $[\mathbf{X_i}, \mathbf{Y_j}] = 0$ otherwise.

Solution-III:

The spacetime coefficients are

$$\nu(x) = 2\ln\left(\frac{x}{\alpha}\right), \quad \mu(x) = b\ln\left(\frac{x}{\alpha}\right),$$

and the values of the functions ξ, η^i and A are

$$\begin{aligned} &\xi = c_1 s + c_2, \quad \eta^0 = c_6, \quad \eta^1 = c_1\frac{x}{2}, \quad \eta^2 = c_1\frac{2-a}{4}y - c_3 z + c_4, \\ &\eta^3 = c_1\frac{2-a}{4}z + c_3 y + c_5, \quad A = c_7. \end{aligned}$$

The spacetime takes the form

$$ds^2 = \left(\frac{x}{\alpha}\right)^2 dt^2 - dx^2 - \left(\frac{x}{\alpha}\right)^b (dy^2 + dz^2), \quad b \neq 0, 2, \quad \alpha \neq 0. \tag{3.5.5}$$

The additional symmetry to the set given in equations (3.4.1) is

$$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{x}{2}\frac{\partial}{\partial x} + \left(\frac{2-b}{4}\right)y\frac{\partial}{\partial y} + \left(\frac{2-b}{4}\right)z\frac{\partial}{\partial z}.$$

It is a homothetic vector. The corresponding first integral is

$$\phi_5 = s\mathcal{L} + x\dot{x} + \frac{2-b}{2}\left(\frac{x}{\alpha}\right)^b (y\dot{y} + z\dot{z}).$$

The Lie algebra of $\mathbf{Y_1}$ with the set of Noether symmetries given in equation (3.4.1) is

$[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_1}, \mathbf{Y_1}] = \frac{2-a}{4}\mathbf{X_1}$,

$[\mathbf{X_2}, \mathbf{Y_1}] = \frac{2-b}{4}\mathbf{X_2}$, $[\mathbf{Y_0}, \mathbf{Y_1}] = \mathbf{Y_0}$, $[\mathbf{X_i}, \mathbf{X_j}] = 0$ and $[\mathbf{X_i}, \mathbf{Y_j}] = 0$ otherwise.

Solution-IV:

Coefficients of the spacetime are

$$\nu(x) = a\ln\left(\frac{x}{\alpha}\right), \quad \mu(x) = 2\ln\left(\frac{x}{\alpha}\right).$$

Components of the Noether symmetry generators are

$$\begin{aligned} &\xi = c_1 s + c_2, \quad \eta^0 = c_1\frac{2-a}{4}t + c_6, \quad \eta^1 = c_1\frac{x}{2}, \\ &\eta^2 = -c_3 z + c_4, \quad \eta^3 = c_3 y + c_5, \quad A = c_7. \end{aligned}$$

The spacetime in this case is

$$ds^2 = \left(\frac{x}{\alpha}\right)^a dt^2 - dx^2 - \left(\frac{x}{\alpha}\right)^2 (dy^2 + dz^2), \quad a \neq 0, 2, \quad \alpha \neq 0. \tag{3.5.6}$$

The additional Noether symmetry is

$$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{x}{2}\frac{\partial}{\partial x} + \left(\frac{2-a}{4}\right) t\frac{\partial}{\partial t}.$$

It is a homothetic vector. The first integral corresponding to the symmetry generator $\mathbf{Y_1}$ is

$$\phi_5 = s\mathcal{L} + x\dot{x} - \left(\frac{2-a}{2}\right)\left(\frac{x}{\alpha}\right)^a t\dot{t}.$$

The Lie algebra is

$[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_0}, \mathbf{Y_1}] = \frac{2-a}{4}\mathbf{X_0}$,

$[\mathbf{Y_0}, \mathbf{Y_1}] = \mathbf{Y_0}$, $[\mathbf{X_i}, \mathbf{X_j}] = 0$ and $[\mathbf{X_i}, \mathbf{Y_j}] = 0$ otherwise.

Solution-V:

The spacetime coefficients are

$$\nu(x) = b\ln\left(\frac{x}{\alpha}\right), \quad \mu(x) = a\ln\left(\frac{x}{\alpha}\right).$$

Components of the Noether symmetry generators are

$$\xi = c_1 s + c_2, \quad \eta^0 = c_1\frac{2-b}{4}t + c_6, \quad \eta^1 = c_1\frac{x}{2},$$
$$\eta^2 = c_1\frac{2-a}{4}y - c_3 z + c_4, \quad \eta^3 = c_1\frac{2-a}{4}z + c_3 y + c_5, \quad A = c_7.$$

We have the following spacetime

$$ds^2 = (\frac{x}{\alpha})^b dt^2 - dx^2 - (\frac{x}{\alpha})^a (dy^2 + dz^2), \quad 2 \neq a \neq b \neq 2, \quad \alpha \neq 0. \tag{3.5.7}$$

The following symmetry along with equations (3.4.1) form a six dimensional algebra

$$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{x}{2}\frac{\partial}{\partial x} + \left(\frac{2-b}{4}\right) t\frac{\partial}{\partial t} + \left(\frac{2-a}{4}\right) y\frac{\partial}{\partial y} + \left(\frac{2-a}{4}\right) z\frac{\partial}{\partial z}. \tag{3.5.8}$$

It is a homothety. The first integral corresponding to Noether symmetry given in equation (3.5.8) is

$$\phi_5 = s\mathcal{L} + x\dot{x} - \frac{2-b}{2}\left(\frac{x}{\alpha}\right)^a t\dot{t} + \frac{2-a}{2}\left(\frac{x}{\alpha}\right)^a (y\dot{y} + z\dot{z}).$$

The Lie algebra is

$[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_0}, \mathbf{Y_1}] = \frac{2-a}{4}\mathbf{X_0}$, $[\mathbf{X_1}, \mathbf{Y_1}] = \frac{2-b}{4}\mathbf{X_1}$,

$[\mathbf{X_2}, \mathbf{Y_1}] = \frac{2-b}{4}\mathbf{X_2}$, $[\mathbf{Y_0}, \mathbf{Y_1}] = \mathbf{Y_0}$, $[\mathbf{X_i}, \mathbf{X_j}] = 0$ and $[\mathbf{X_i}, \mathbf{Y_j}] = 0$ otherwise.

3.6 Seven Noether Symmetries and First Integrals

Solutions of the system given in equations (3.3.1) for which we get seven Noether symmetries are given in this section:

Solution-I:

The spacetime coefficients are

$$\nu(x) = \mu(x), \quad \nu(x) \neq a \ln\left(\frac{x}{\alpha}\right), \quad a \neq 0 \neq \alpha.$$

The values of the functions ξ, η^i and A are

$$\xi = c_1, \quad \eta^0 = c_2 y + c_3 z + c_4, \quad \eta^1 = 0, \quad \eta^2 = c_2 t - c_5 z + c_6,$$
$$\eta^3 = c_3 t + c_5 y + c_7, \quad A = c_8.$$

Generally the spacetime takes the form

$$ds^2 = e^{\nu(x)} dt^2 - dx^2 - e^{\nu(x)}(dy^2 + dz^2), \tag{3.6.1}$$

some examples are

$$(i): \quad ds^2 = \cosh^2\left(\frac{x}{\alpha}\right) dt^2 - dx^2 - \cosh^2\left(\frac{x}{\alpha}\right)(dy^2 + dz^2), \quad \alpha \neq 0, \tag{3.6.2}$$

$$(ii): \quad ds^2 = \cos^2\left(\frac{x}{\alpha}\right) dt^2 - dx^2 - \cos^2\left(\frac{x}{\alpha}\right)(dy^2 + dz^2), \quad \alpha \neq 0. \tag{3.6.3}$$

The additional symmetries are

$$\mathbf{X_4} = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, \quad \mathbf{X_5} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}. \tag{3.6.4}$$

Both $\mathbf{X_4}$ and $\mathbf{X_5}$ are isometries. The first integrals are given in Table 3.3.

Table 3.3: First Integrals

Gen	First Integrals
$\mathbf{X_4}(i)$	$\phi_5 = 2\cosh^2\left(\frac{x}{\alpha}\right)(\dot{y}t - y\dot{t})$
$\mathbf{X_5}(i)$	$\phi_6 = 2\cosh^2\left(\frac{x}{\alpha}\right)(\dot{z}t - z\dot{t})$
$\mathbf{X_4}(ii)$	$\phi_7 = 2\cos^2\left(\frac{x}{\alpha}\right)(\dot{y}t - y\dot{t})$
$\mathbf{X_5}(ii)$	$\phi_8 = 2\cos^2\left(\frac{x}{\alpha}\right)(\dot{z}t - z\dot{t})$

The Lie algebra of $\mathbf{X_4}$ and $\mathbf{X_5}$ along with the symmetries of equation (3.4.1) is

$[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_0}, \mathbf{X_4}] = \mathbf{X_1}$, $[\mathbf{X_1}, \mathbf{X_4}] = \mathbf{X_0}$, $[\mathbf{X_3}, \mathbf{X_4}] = -\mathbf{X_5}$,

$[\mathbf{X_0}, \mathbf{X_5}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_5}] = \mathbf{X_0}$, $[\mathbf{X_3}, \mathbf{X_5}] = \mathbf{X_4}$, $[\mathbf{X_4}, \mathbf{X_5}] = \mathbf{X_3}$,

$[\mathbf{X_i}, \mathbf{X_j}] = 0$ and $[\mathbf{X_i}, \mathbf{Y_0}] = 0$ otherwise.

Solution-II:

The metric coefficients are

$$\nu(x) = c, \quad \mu(x) = a \ln\left(\frac{x}{\alpha}\right).$$

Components of the Noether symmetry generators are

$$\xi = c_1 s + c_2, \quad \eta^0 = c_1 \frac{t}{2} + c_3 s + c_4, \quad \eta^1 = c_1 \frac{x}{2},$$
$$\eta^2 = c_1 \frac{2-a}{4} y - c_5 z + c_6, \quad \eta^3 = c_1 \frac{2-a}{4} z + c_5 y + c_7, \quad A = c_8.$$

The spacetime takes the form

$$ds^2 = dt^2 - dx^2 - \left(\frac{x}{\alpha}\right)^a (dy^2 + dz^2), \quad 2 \neq a \neq 0 \neq \alpha. \tag{3.6.5}$$

The symmetries are

$$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{t}{2}\frac{\partial}{\partial t} + \left(\frac{2-a}{4}\right) y \frac{\partial}{\partial y} + \left(\frac{2-a}{4}\right) z \frac{\partial}{\partial z}, \quad \mathbf{Y_2} = s\frac{\partial}{\partial t}, \quad A = 2t. \tag{3.6.6}$$

$\mathbf{Y_1}$ is homothety and $\mathbf{Y_2}$ is neither homothety nor isometry. The first integrals are given in Table 3.4.

Table 3.4: First Integrals

Gen	First Integrals
$\mathbf{Y_1}$	$\phi_5 = s\mathcal{L} + x\dot{x} + t\dot{t} + \frac{2-a}{2}\left(\frac{x}{\alpha}\right)^a (y\dot{y} + z\dot{z})$
$\mathbf{Y_2}$	$\phi_6 = 2(t - s\dot{t})$

The Lie algebra of $\mathbf{Y_1}$ and $\mathbf{Y_2}$ and equation (3.4.1) is

$[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_0}, \mathbf{Y_1}] = \frac{1}{2}\mathbf{X_0}$, $[\mathbf{X_1}, \mathbf{Y_1}] = \frac{2-a}{4}\mathbf{X_1}$, $[\mathbf{X_2}, \mathbf{Y_1}] = \frac{2-a}{4}\mathbf{X_2}$,

$[\mathbf{Y_0}, \mathbf{Y_1}] = \mathbf{Y_0}$, $[\mathbf{Y_1}, \mathbf{Y_2}] = \frac{1}{2}\mathbf{Y_2}$, $[\mathbf{X_i}, \mathbf{X_j}] = 0$ and $[\mathbf{X_i}, \mathbf{Y_j}] = 0$ otherwise.

Solution-III:

Values of $\nu(x)$ and $\mu(x)$ are

$$\nu = \nu(x), \quad \mu(x) = c.$$

Components of the Noether symmetry generators are

$$\xi = c_1, \quad \eta^0 = c_2, \quad \eta^1 = 0, \quad \eta^2 = c_3 s - c_4 z + c_5,$$
$$\eta^3 = c_6 s + c_4 y + c_7, \quad A = c_8.$$

Here the spacetime is

$$ds^2 = e^{\nu(x)} dt^2 - dx^2 - (dy^2 + dz^2), \quad \nu(x) \neq a \ln\left(\frac{x}{\alpha}\right), \quad a \neq 0 \neq \alpha, \quad \nu''(x) \neq 0. \quad (3.6.7)$$

The additional symmetries are

$$\mathbf{Y_1} = s\frac{\partial}{\partial y}, \quad A_1 = -2y, \quad \mathbf{Y_2} = s\frac{\partial}{\partial z}, \quad A_2 = -2z. \quad (3.6.8)$$

Both $\mathbf{Y_1}$ and $\mathbf{Y_2}$ are neither homothety nor isometry. The corresponding first integrals are given in Table 3.5.

Table 3.5: First Integrals

Gen	First Integrals
$\mathbf{Y_1}$	$\phi_5 = 2(s\dot{y} - y)$
$\mathbf{Y_2}$	$\phi_6 = 2(s\dot{z} - z)$

The Lie algebra of the Noether symmetry generators is

$[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{Y_0}, \mathbf{Y_1}] = \mathbf{X_1}$, $[\mathbf{Y_0}, \mathbf{Y_2}] = \mathbf{X_2}$,

$[\mathbf{X_i}, \mathbf{X_j}] = 0$, $[\mathbf{X_i}, \mathbf{Y_j}] = 0$ and $[\mathbf{Y_i}, \mathbf{Y_j}] = 0$ otherwise.

3.7 Eight Noether Symmetries and First Integrals

In this section all those solutions are given where the system given by equations (3.3.1) gives eight Noether symmetries:

Solution-I:

The metric coefficients in this case are

$$\nu(x) = a \ln\left(\frac{x}{\alpha}\right) = \mu(x).$$

Components of the Noether symmetry generators are

$$\xi = c_1 s + c_2, \quad \eta^0 = c_1\frac{2-a}{4}t + c_3 y + c_4 z + c_5, \quad \eta^1 = c_1\frac{x}{2},$$
$$\eta^2 = c_3 t + c_1\frac{2-a}{4}y - c_6 z + c_7, \quad \eta^3 = c_4 t + c_1\frac{2-a}{4}z + c_6 y + c_8, \quad A = c_9.$$

We get the following spacetime here

$$ds^2 = \left(\frac{x}{\alpha}\right)^a dt^2 - dx^2 - \left(\frac{x}{\alpha}\right)^a (dy^2 + dz^2), \quad a \neq 0, 2 \quad \alpha \neq 0. \tag{3.7.1}$$

The additional Noether symmetries are

$$\mathbf{X_4} = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, \quad \mathbf{X_5} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \tag{3.7.2}$$

$$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{x}{2}\frac{\partial}{\partial x} + \left(\frac{2-a}{4}\right)t\frac{\partial}{\partial t} + \left(\frac{2-a}{4}\right)y\frac{\partial}{\partial y} + \left(\frac{2-a}{4}\right)z\frac{\partial}{\partial z}. \tag{3.7.3}$$

$\mathbf{X_4}$, $\mathbf{X_5}$ are isometries and $\mathbf{Y_1}$ is homothety. The first integrals are given in Table 3.6.

Table 3.6: First Integrals

Gen	First Integrals
$\mathbf{X_4}$	$\phi_5 = 2\left(\frac{x}{\alpha}\right)^a(\dot{y}t - y\dot{t})$
$\mathbf{X_5}$	$\phi_6 = 2\left(\frac{x}{\alpha}\right)^a(\dot{z}t - z\dot{t})$
$\mathbf{Y_1}$	$\phi_7 = s\mathcal{L} + x\dot{x} - \frac{2-a}{2}\left(\frac{x}{\alpha}\right)^a t\dot{t} + \frac{2-a}{2}\left(\frac{x}{\alpha}\right)^a(y\dot{y} + z\dot{z})$

The Lie algebra is

$[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_0}, \mathbf{X_4}] = \mathbf{X_1}$, $[\mathbf{X_1}, \mathbf{X_4}] = \mathbf{X_0}$, $[\mathbf{X_3}, \mathbf{X_4}] = -\mathbf{X_5}$, $[\mathbf{X_0}, \mathbf{X_5}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_5}] = \mathbf{X_0}$, $[\mathbf{X_3}, \mathbf{X_5}] = \mathbf{X_4}$, $[\mathbf{X_4}, \mathbf{X_5}] = \mathbf{X_3}$, $[\mathbf{X_0}, \mathbf{Y_2}] = \frac{2-a}{4}\mathbf{X_0}$, $[\mathbf{X_1}, \mathbf{Y_1}] = \frac{2-a}{4}\mathbf{X_1}$, $[\mathbf{X_2}, \mathbf{Y_1}] = \frac{2-a}{4}\mathbf{X_2}$, $[\mathbf{Y_0}, \mathbf{Y_1}] = \mathbf{Y_0}$, $[\mathbf{X_i}, \mathbf{X_j}] = 0$, $[\mathbf{X_i}, \mathbf{Y_j}] = 0$ and $[\mathbf{Y_i}, \mathbf{Y_j}] = 0$ otherwise.

Solution-II:

Coefficients of the spacetime are

$$\nu(x) = c, \quad \mu(x) = 2\ln\left(\frac{x}{\alpha}\right).$$

The values of the functions ξ, η^i and A are

$$\xi = c_1 s^2 + c_2 s + c_3, \quad \eta^0 = c_1 ts + c_2\frac{t}{2} + c_8 s + c_4, \quad \eta^1 = c_1\frac{xs}{2} + c_2\frac{x}{2},$$
$$\eta^2 = -c_5 z + c_6, \quad \eta^3 = c_5 y + c_7, \quad A = c_1 t^2 - x^2 + c_9.$$

The spacetime takes the form

$$ds^2 = dt^2 - dx^2 - \left(\frac{x}{\alpha}\right)^2 (dy^2 + dz^2), \quad \alpha \neq 0, \tag{3.7.4}$$

and the additional Noether symmetry generators are

$$\mathbf{Y_1} = s\frac{\partial}{\partial t}, \quad A_1 = 2t, \quad \mathbf{Y_2} = s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x},$$
$$\mathbf{Y_3} = s^2\frac{\partial}{\partial s} + st\frac{\partial}{\partial t} + sx\frac{\partial}{\partial x}, \quad A_3 = t^2 - x^2.$$

$\mathbf{Y_2}$ is homothety and $\mathbf{Y_1}$, $\mathbf{Y_3}$ are neither homothety nor isometry. Table 3.7 contains the corresponding first integrals.

Table 3.7: First Integrals

Gen	First Integrals
$\mathbf{Y_1}$	$\phi_5 = 2(t - s\dot{t})$
$\mathbf{Y_2}$	$\phi_6 = s\mathcal{L} + x\dot{x} - t\dot{t}$
$\mathbf{Y_3}$	$\phi_7 = s^2\mathcal{L} + 2sx\dot{x} - 2st\dot{t} + t^2 - x^2$

The Lie algebra of the Noether symmetries is

$[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{Y_0}, \mathbf{Y_1}] = \mathbf{X_0}$, $[\mathbf{X_0}, \mathbf{Y_2}] = \frac{1}{2}\mathbf{X_0}$, $[\mathbf{Y_0}, \mathbf{Y_2}] = \mathbf{Y_0}$,
$[\mathbf{X_0}, \mathbf{Y_3}] = \mathbf{Y_1}$, $[\mathbf{Y_0}, \mathbf{Y_3}] = 2\mathbf{Y_2}$, $[\mathbf{Y_1}, \mathbf{Y_2}] = -\frac{1}{2}\mathbf{Y_1}$, $[\mathbf{Y_2}, \mathbf{Y_3}] = \mathbf{Y_3}$,
$[\mathbf{X_i}, \mathbf{X_j}] = 0$, $[\mathbf{X_i}, \mathbf{Y_j}] = 0$ and $[\mathbf{Y_i}, \mathbf{Y_j}] = 0$ otherwise.

Solution-III:

Coefficients of the metric are

$$\nu(x) = a\ln\left(\frac{x}{\alpha}\right), \quad \mu(x) = c.$$

Components of the Noether symmetry generators are

$$\xi = c_1 s + c_2, \quad \eta^0 = c_1\frac{2-a}{4}t + c_3, \quad \eta^1 = c_1\frac{x}{2},$$
$$\eta^2 = c_1\frac{y}{2} + c_4 s - c_5 z + c_6, \quad \eta^3 = c_1\frac{z}{2} + c_7 s + c_5 y + c_8, \quad A = -2c_4 y - 2c_7 z + c_9.$$

The corresponding metric is

$$ds^2 = \left(\frac{x}{\alpha}\right)^a dt^2 - dx^2 - (dy^2 + dz^2), \qquad a \neq 0, 2, \quad \alpha \neq 0. \tag{3.7.5}$$

The symmetries other than the minimal set are,

$$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \left(\frac{2-a}{4}\right)t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}, \tag{3.7.6}$$

$$\mathbf{Y_2} = s\frac{\partial}{\partial y}, \quad A_2 = -2y, \quad \mathbf{Y_3} = s\frac{\partial}{\partial z}, \quad A_3 = -2z. \tag{3.7.7}$$

$\mathbf{Y_1}$ is homothetic vector while $\mathbf{Y_2}$ and $\mathbf{Y_3}$ are neither homothety nor isometry. The first integrals are given in the following table.

Table 3.8: First Integrals

Gen	First Integrals
$\mathbf{Y_1}$	$\phi_5 = s\mathcal{L} + x\dot{x} - \left(\frac{2-a}{2}\right)\left(\frac{x}{\alpha}\right)^a t\dot{t} + (y\dot{y} + z\dot{z})$
$\mathbf{Y_2}$	$\phi_6 = 2(s\dot{y} - y)$
$\mathbf{Y_3}$	$\phi_7 = 2(s\dot{z} - z)$

The Lie algebra of $\mathbf{Y_1}$, $\mathbf{Y_2}$ and $\mathbf{Y_3}$ along with the symmetries of equation (3.4.1) is $[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{Y_0}, \mathbf{Y_2}] = \mathbf{X_1}$, $[\mathbf{Y_0}, \mathbf{Y_3}] = \mathbf{X_2}$, $[\mathbf{Y_2}, \mathbf{Y_1}] = \frac{-1}{2}\mathbf{Y_2}$, $[\mathbf{Y_1}, \mathbf{Y_3}] = \frac{-1}{2}\mathbf{Y_3}$, $[\mathbf{X_0}, \mathbf{Y_1}] = \frac{2-a}{4}\mathbf{X_0}$, $[\mathbf{X_1}, \mathbf{Y_1}] = \frac{1}{2}\mathbf{X_1}$, $[\mathbf{X_2}, \mathbf{Y_1}] = \frac{1}{2}\mathbf{X_2}$, $[\mathbf{Y_0}, \mathbf{Y_1}] = \mathbf{Y_0}$, $[\mathbf{X_i}, \mathbf{X_j}] = 0$ and $[\mathbf{X_i}, \mathbf{Y_j}] = 0$ otherwise.

3.8 Nine Noether Symmetries and First Integrals

We have the following solutions with nine Noether symmetries:

Solution-I:

Coefficients of the spacetime are

$$\nu(x) = c, \quad \mu(x) = \frac{x}{\alpha}.$$

Values of the functions ξ, η^i, and A are

$$\xi = c_1, \quad \eta^0 = c_2 s + c_3, \quad \eta^1 = c_4 y + c_5 z + c_6,$$
$$\eta^2 = -c_6\left(\frac{y}{2\alpha}\right) + c_4\left(\frac{-y^2}{4\alpha} + \frac{z^2}{4a} + \alpha e^{\frac{-x}{\alpha}}\right) - c_5\left(\frac{yz}{2\alpha}\right) - c_7 z + c_8,$$
$$\eta^3 = -c_6\left(\frac{z}{2\alpha}\right) + c_5\left(\frac{y^2}{4\alpha} - \frac{z^2}{4\alpha} + \alpha e^{\frac{-x}{\alpha}}\right) - c_4\left(\frac{yz}{2a}\right) + c_7 y + c_9, \quad A = 2c_2 t + c_{10}.$$

The metric of the spacetime is

$$ds^2 = dt^2 - dx^2 - e^{\frac{x}{\alpha}}(dy^2 + dz^2), \quad \alpha \neq 0. \tag{3.8.1}$$

The additional Noether symmetry generators are

$$\mathbf{X_4} = \frac{\partial}{\partial x} - \frac{y}{2\alpha}\frac{\partial}{\partial y} - \frac{z}{2\alpha}\frac{\partial}{\partial z}, \quad \mathbf{X_5} = y\frac{\partial}{\partial x} + \left(-\frac{y^2}{4\alpha} + \frac{z^2}{4\alpha} + ae^{-\frac{x}{\alpha}}\right)\frac{\partial}{\partial y} - \frac{yz}{2\alpha}\frac{\partial}{\partial z}, \tag{3.8.2}$$

$$\mathbf{X_6} = z\frac{\partial}{\partial x} + \left(-\frac{z^2}{4\alpha} + \frac{y^2}{4\alpha} + ae^{-\frac{x}{\alpha}}\right)\frac{\partial}{\partial z} - \frac{yz}{2\alpha}\frac{\partial}{\partial y}, \quad \mathbf{Y_1} = s\frac{\partial}{\partial t}, \quad A_1 = 2t. \tag{3.8.3}$$

$\mathbf{X_4}$, $\mathbf{X_5}$ and $\mathbf{X_6}$ are isometries. The first integrals are given in Table 3.9.

Table 3.9: First Integrals

Gen	First Integrals
$\mathbf{X_4}$	$\phi_5 = 2\dot{x} - \frac{e^{\frac{x}{\alpha}}}{\alpha}(y\dot{y} + z\dot{z})$
$\mathbf{X_5}$	$\phi_6 = 2\dot{x}y + \frac{e^{\frac{x}{\alpha}}}{2\alpha}[\dot{y}(z^2 - y^2 + 4\alpha^2 e^{\frac{-x}{\alpha}}) - yz\dot{z}]$
$\mathbf{X_6}$	$\phi_7 = 2\dot{x}z + \frac{e^{\frac{x}{\alpha}}}{2\alpha}[\dot{z}(-z^2 + y^2 + 4\alpha^2 e^{\frac{-x}{\alpha}}) - yz\dot{y}]$
$\mathbf{Y_1}$	$\phi_8 = 2(t - s\dot{t})$

The Lie algebra of $\mathbf{X_4}$, $\mathbf{X_5}$, $\mathbf{X_6}$ and $\mathbf{Y_1}$ along with the generators of equation (3.4.1) is $[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_1}, \mathbf{X_4}] = \frac{-1}{2\alpha}\mathbf{X_1}$, $[\mathbf{X_1}, \mathbf{X_4}] = \frac{-1}{2\alpha}\mathbf{X_2}$, $[\mathbf{X_1}, \mathbf{X_5}] = \mathbf{X_4}$, $[\mathbf{X_2}, \mathbf{X_5}] = \frac{-1}{2\alpha}\mathbf{X_3}$, $[\mathbf{X_3}, \mathbf{X_5}] = -\mathbf{X_6}$, $[\mathbf{X_1}, \mathbf{X_6}] = \frac{1}{2\alpha}\mathbf{X_3}$, $[\mathbf{X_2}, \mathbf{X_6}] = \mathbf{X_4}$, $[\mathbf{X_3}, \mathbf{X_6}] = \mathbf{X_5}$, $[\mathbf{X_4}, \mathbf{X_6}] = \frac{-1}{2\alpha}\mathbf{X_5}$, $[\mathbf{X_4}, \mathbf{X_5}] = \frac{-1}{2\alpha}\mathbf{X_6}$, $[\mathbf{Y_0}, \mathbf{Y_1}] = \mathbf{X_0}$, $[\mathbf{X_i}, \mathbf{X_j}] = 0$, $[\mathbf{X_i}, \mathbf{Y_j}] = 0$ and $[\mathbf{Y_i}, \mathbf{Y_j}] = 0$ otherwise.

Solution-II:

Values of the fuctions $\nu(x)$ and $\mu(x)$ are

$$\nu(x) = \frac{x}{\alpha}, \quad \mu(x) = c.$$

Components of the Noether symmetry generators are

$$\xi = c_1, \quad \eta^0 = -c_2\frac{t}{2\alpha} - c_3\left(\frac{t^2}{4\alpha} + \alpha e^{\frac{-x}{\alpha}}\right) + c_4, \quad \eta^1 = c_2 + c_3 t,$$

$$\eta^2 = c_5 s - c_6 z + c_7, \quad \eta^3 = c_8 s + c_6 y + c_9, \quad A = c_5 y + c_8 z + c_{10}.$$

The spacetime takes the form

$$ds^2 = e^{\frac{x}{\alpha}}dt^2 - dx^2 - (dy^2 + dz^2), \quad \alpha \neq 0. \tag{3.8.4}$$

The symmetries other than given by equations (3.4.1) are

$$\mathbf{X_4} = \frac{\partial}{\partial x} - \frac{t}{2\alpha}\frac{\partial}{\partial t}, \quad \mathbf{X_5} = t\frac{\partial}{\partial x} - \left(\frac{t^2}{4\alpha} + \alpha e^{-\frac{x}{\alpha}}\right)\frac{\partial}{\partial t},$$
$$\mathbf{Y_1} = s\frac{\partial}{\partial y}, \quad A_1 = -2y, \quad \mathbf{Y_2} = s\frac{\partial}{\partial z}, \quad A_1 = -2z.$$

$\mathbf{X_4}$, $\mathbf{X_5}$ are isometries. The first integrals are given in Table 3.10.

Table 3.10: First Integrals

Gen	First Integrals
$\mathbf{X_4}$	$\phi_5 = \frac{tte^{\frac{x}{\alpha}}}{\alpha} + 2\dot{x}$
$\mathbf{X_5}$	$\phi_6 = 2\dot{x}t + (t^2 e^{\frac{x}{\alpha}} + 4\alpha^2)\frac{\dot{t}}{2\alpha}$
$\mathbf{Y_1}$	$\phi_7 = 2(s\dot{y} - y)$
$\mathbf{Y_2}$	$\phi_8 = 2(s\dot{z} - z)$

The Lie algebra of $\mathbf{X_4}$, $\mathbf{X_5}$, $\mathbf{Y_1}$, and $\mathbf{Y_2}$ and symmetries of equation (3.4.1) is
$[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_0}, \mathbf{X_4}] = \frac{-1}{2\alpha}\mathbf{X_0}$, $[\mathbf{X_0}, \mathbf{X_5}] = \mathbf{X_4}$,
$[\mathbf{X_2}, \mathbf{Y_1}] = \mathbf{X_4}$, $[\mathbf{Y_0}, \mathbf{Y_1}] = \mathbf{X_1}$, $[\mathbf{Y_0}, \mathbf{Y_2}] = \mathbf{X_2}$, $[\mathbf{X_4}, \mathbf{X_5}] = \frac{-1}{2\alpha}\mathbf{X_5}$,
$[\mathbf{X_i}, \mathbf{X_j}] = 0$, $[\mathbf{X_i}, \mathbf{Y_j}] = 0$ and $[\mathbf{Y_i}, \mathbf{Y_j}] = 0$ otherwise.

Solution-III:

The metric coefficients are

$$\nu(x) = 2\ln\cosh\left(\frac{x}{\alpha}\right), \quad \mu(x) = c.$$

The values of ξ, η^i and A are

$$\xi = c_1, \quad \eta^0 = -c_2\tanh\left(\frac{x}{\alpha}\right)\sin\left(\frac{t}{\alpha}\right) + c_3\tanh\left(\frac{x}{\alpha}\right)\cos\left(\frac{t}{\alpha}\right) + c_4,$$
$$\eta^1 = c_2\cos\left(\frac{t}{\alpha}\right) + c_3\sin\left(\frac{t}{\alpha}\right), \quad \eta^2 = c_5 s - c_6 z + c_7,$$
$$\eta^3 = c_8 s + c_6 y + c_9, \quad A = c_5 y + c_8 z + c_{10}.$$

The metric in this case is

$$ds^2 = \cosh^2\left(\frac{x}{\alpha}\right)dt^2 - dx^2 - (dy^2 + dz^2), \quad \alpha \neq 0. \tag{3.8.5}$$

The additional Noether symmetry generators are

$$\mathbf{X_4} = -\tanh\left(\frac{x}{\alpha}\right)\sin\left(\frac{t}{\alpha}\right)\frac{\partial}{\partial t} + \cos\left(\frac{t}{\alpha}\right)\frac{\partial}{\partial x},$$
$$\mathbf{X_5} = \tanh\left(\frac{x}{\alpha}\right)\cos\left(\frac{t}{\alpha}\right)\frac{\partial}{\partial t} + \sin\left(\frac{t}{\alpha}\right)\frac{\partial}{\partial x}, \tag{3.8.6}$$

$$\mathbf{Y_1} = s\frac{\partial}{\partial y}, \quad A_1 = -2y, \quad \mathbf{Y_2} = s\frac{\partial}{\partial z}, \quad A_2 = -2z. \tag{3.8.7}$$

$\mathbf{X_4}$, $\mathbf{X_5}$ are isometries. Table 3.11 contains the corresponding first integrals.

Table 3.11: First Integrals

Gen	First Integrals
$\mathbf{X_4}$	$\phi_5 = 2\left(\dot{t}\sinh\left(\frac{x}{\alpha}\right)\sin\left(\frac{t}{\alpha}\right)\cosh\left(\frac{x}{\alpha}\right) + \dot{x}\cos\left(\frac{t}{\alpha}\right)\right)$
$\mathbf{X_5}$	$\phi_6 = 2\left(-\dot{t}\sinh\left(\frac{x}{\alpha}\right)\cos\left(\frac{t}{\alpha}\right)\cosh\left(\frac{x}{\alpha}\right) + \dot{x}\sin\left(\frac{t}{\alpha}\right)\right)$
$\mathbf{Y_1}$	$\phi_7 = 2(s\dot{y} - y)$
$\mathbf{Y_2}$	$\phi_8 = 2(s\dot{z} - z)$

The Lie Algebra for this case is

$[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_0}, \mathbf{X_4}] = \frac{1}{\alpha}\mathbf{X_5}$, $[\mathbf{X_0}, \mathbf{X_5}] = \frac{1}{a}\mathbf{X_4}$, $[\mathbf{Y_0}, \mathbf{Y_1}] = \mathbf{X_1}$,

$[\mathbf{Y_0}, \mathbf{Y_2}] = \mathbf{X_2}$, $[\mathbf{X_i}, \mathbf{X_j}] = 0$, $[\mathbf{X_i}, \mathbf{Y_j}] = 0$,and $[\mathbf{Y_i}, \mathbf{Y_j}] = 0$ otherwise.

Solution-IV:

Coefficients of the spacetime are

$$\nu(x) = 2\ln\cos\left(\frac{x}{\alpha}\right), \quad \mu(x) = c.$$

Components of the Noether symmetry generators are

$$\xi = c_1, \quad \eta^0 = -c_2\tan\left(\frac{x}{\alpha}\right)\sin\left(\frac{t}{\alpha}\right) + c_3\tan\left(\frac{x}{\alpha}\right)\cos\left(\frac{t}{\alpha}\right) + c_4,$$
$$\eta^1 = c_2\cos\left(\frac{t}{\alpha}\right) + c_3\sin\left(\frac{t}{\alpha}\right), \quad \eta^2 = c_5 s - c_6 z + c_7,$$
$$\eta^3 = c_8 s + c_6 y + c_9, \quad A = c_5 y + c_8 z + c_{10}.$$

The spacetime takes the from

$$ds^2 = \cos^2\left(\frac{x}{\alpha}\right)dt^2 - dx^2 - (dy^2 + dz^2), \quad \alpha \neq 0. \tag{3.8.8}$$

The additional symmetries are

$$\begin{aligned} \mathbf{X_4} &= -\tan\left(\frac{x}{\alpha}\right)\sin\left(\frac{t}{\alpha}\right)\frac{\partial}{\partial t} + \cos\left(\frac{t}{\alpha}\right)\frac{\partial}{\partial x}, \\ \mathbf{X_5} &= \tan\left(\frac{x}{\alpha}\right)\cos\left(\frac{t}{\alpha}\right)\frac{\partial}{\partial t} + \sin\left(\frac{t}{\alpha}\right)\frac{\partial}{\partial x}, \end{aligned} \tag{3.8.9}$$

$$\mathbf{Y_1} = s\frac{\partial}{\partial y}, \quad A_1 = -2y, \quad \mathbf{Y_2} = s\frac{\partial}{\partial z}, \quad A_2 = -2z. \tag{3.8.10}$$

$\mathbf{X_4}$, $\mathbf{X_5}$ are isometries. The first integrals are given in Table 3.12.

Table 3.12: First Integrals

Gen	First Integrals
$\mathbf{X_4}$	$\phi_5 = 2\left(\dot{t}\sin\left(\frac{x}{\alpha}\right)\sin\left(\frac{t}{\alpha}\right)\cos\left(\frac{x}{\alpha}\right) + \dot{x}\cos\left(\frac{t}{\alpha}\right)\right)$
$\mathbf{X_5}$	$\phi_6 = 2\left(-\dot{t}\sin\left(\frac{x}{\alpha}\right)\cos\left(\frac{t}{\alpha}\right)\cos\left(\frac{x}{\alpha}\right) + \dot{x}\sin\left(\frac{t}{\alpha}\right)\right)$
$\mathbf{Y_1}$	$\phi_7 = 2(s\dot{y} - y)$
$\mathbf{Y_2}$	$\phi_8 = 2(s\dot{z} - z)$

The Lie algebra of $\mathbf{X_4}$, $\mathbf{X_5}$, $\mathbf{Y_1}$ and $\mathbf{Y_2}$ along with the generators of equation (3.4.1) is $[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_0}, \mathbf{X_4}] = \frac{-1}{\alpha}\mathbf{X_5}$, $[\mathbf{X_0}, \mathbf{X_5}] = \frac{1}{\alpha}\mathbf{X_4}$, $[\mathbf{Y_0}, \mathbf{Y_1}] = \mathbf{X_1}$, $[\mathbf{Y_0}, \mathbf{Y_2}] = \mathbf{X_2}, [\mathbf{X_i}, \mathbf{X_j}] = 0$, $[\mathbf{X_i}, \mathbf{Y_j}] = 0$ and $[\mathbf{Y_i}, \mathbf{Y_j}] = 0$ otherwise.

Solution-V:

Coefficients of the metric are

$$\nu(x) = 2\ln\left(\frac{x}{\alpha}\right) = \mu(x).$$

Components of the Noethter symmetries are

$$\begin{aligned} &\xi = c_1 s^2 + c_2 s + c_3, \quad \eta^0 = c_4 y + c_5 z + c_6, \quad \eta^1 = c_1 x s + c_2\frac{x}{2}, \\ &\eta^2 = c_4 t - c_7 z + c_8, \quad \eta^3 = c_5 t + c_7 y + c_9, \quad A = -c_1 x^2 + c_{10}. \end{aligned}$$

The metric for this solution is

$$ds^2 = \left(\frac{x}{\alpha}\right)^2 dt^2 - dx^2 - \left(\frac{x}{\alpha}\right)^2 (dy^2 + dz^2), \quad \alpha \neq 0. \tag{3.8.11}$$

The symmetries other than given in equations (3.4.1) are

$$\mathbf{X_4} = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, \quad \mathbf{X_5} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \quad \mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{x}{2}\frac{\partial}{\partial x}, \tag{3.8.12}$$

$$\mathbf{Y_2} = s^2\frac{\partial}{\partial s} + sx\frac{\partial}{\partial x}, \quad A = -x^2. \tag{3.8.13}$$

$\mathbf{X_4}$, $\mathbf{X_5}$ are isometries and $\mathbf{Y_1}$ is a homothety. The first integrals are given in Table 3.13.

Table 3.13: First Integrals

Gen	First Integrals
$\mathbf{X_4}$	$\phi_5 = 2\left(\frac{x}{\alpha}\right)^2(\dot{y}t - y\dot{t})$
$\mathbf{X_5}$	$\phi_6 = 2\left(\frac{x}{\alpha}\right)^2(\dot{z}t - z\dot{t})$
$\mathbf{Y_1}$	$\phi_7 = s\mathcal{L} + x\dot{x}$
$\mathbf{Y_2}$	$\phi_8 = s^2\mathcal{L} + 2sx\dot{x} - 2x^2$

The Lie algebra of $\mathbf{X_4}$, $\mathbf{X_5}$, $\mathbf{Y_1}$ and $\mathbf{Y_2}$ along with the generators of equation (3.4.1) is $[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_0}, \mathbf{X_4}] = \mathbf{X_1}$, $[\mathbf{X_3}, \mathbf{X_4}] = -\mathbf{X_5}$, $[\mathbf{X_0}, \mathbf{X_5}] = \mathbf{X_2}$, $[\mathbf{X_3}, \mathbf{X_5}] = \mathbf{X_4}$, $[\mathbf{X_4}, \mathbf{X_5}] = \mathbf{X_3}$, $[\mathbf{Y_0}, \mathbf{Y_5}] = 2\mathbf{Y_2}$, $[\mathbf{Y_2}, \mathbf{Y_2}] = \mathbf{Y_2}$, $[\mathbf{X_i}, \mathbf{X_j}] = 0$, $[\mathbf{X_i}, \mathbf{Y_j}] = 0$ and $[\mathbf{Y_i}, \mathbf{Y_j}] = 0$ otherwise.

3.9 Eleven Noether symmetries and First Integrals

We get only one solution for eleven Noether symmetries:

Solution:

Coefficients of the metric are

$$\nu(x) = \frac{x}{\alpha} = \mu(x).$$

Components of the Noether symmetry generators are

$$\begin{aligned}
&\xi = c_1, \quad \eta^1 = c_2 + c_5 y + c_6 z + c_7 t, \quad A = c_{12},\\
&\eta^0 = -c_2\frac{t}{2\alpha} + c_3 z + c_4 y - c_5\frac{yt}{2\alpha} - c_6\frac{zt}{2\alpha} - c_7\left(\frac{t^2}{4\alpha} + \frac{y^2}{4\alpha} + \frac{z^2}{4\alpha} + \alpha e^{\frac{-x}{\alpha}}\right) + c_8,\\
&\eta^2 = -c_2\frac{y}{2\alpha} + c_4 t - c_6\frac{yz}{2\alpha} - c_7\frac{yt}{2\alpha} - c_5\left(\frac{t^2}{4\alpha} + \frac{y^2}{4\alpha} - \frac{z^2}{4\alpha} - ae^{\frac{-x}{\alpha}}\right) - c_9 z + c_{10},\\
&\eta^3 = -c_2\frac{z}{2\alpha} + c_3 t - c_5\frac{yz}{2\alpha} - c_7\frac{zt}{2\alpha} - c_6\left(\frac{t^2}{4\alpha} - \frac{y^2}{4\alpha} + \frac{z^2}{4\alpha} - \alpha e^{\frac{-x}{\alpha}}\right) + c_9 y + c_{11}.
\end{aligned}$$

The metric of the spacetime takes the form

$$ds^2 = e^{\frac{x}{\alpha}}dt^2 - dx^2 - e^{\frac{x}{\alpha}}(dy^2 + dz^2), \quad \alpha \neq 0, \tag{3.9.1}$$

which is the famous anti de-Sitter spacetime. Symmetries other than the minimal set given in equations (3.4.1) are

$$\mathbf{X_4} = \frac{\partial}{\partial x} - \frac{t}{2\alpha}\frac{\partial}{\partial t} - \frac{y}{2\alpha}\frac{\partial}{\partial y} - \frac{z}{2\alpha}\frac{\partial}{\partial z}, \quad \mathbf{X_5} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \quad \mathbf{X_6} = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y},$$

$$\mathbf{X_7} = y\frac{\partial}{\partial x} - \frac{yt}{2\alpha}\frac{\partial}{\partial t} - \frac{yz}{2\alpha}\frac{\partial}{\partial z} - \left(\frac{t^2}{4\alpha} + \frac{y^2}{4\alpha} - \frac{z^2}{4\alpha} - \alpha e^{-\frac{x}{\alpha}}\right)\frac{\partial}{\partial y},$$

$$\mathbf{X_8} = z\frac{\partial}{\partial x} - \frac{zt}{2\alpha}\frac{\partial}{\partial t} - \frac{yz}{2\alpha}\frac{\partial}{\partial y} - \left(\frac{t^2}{4\alpha} - \frac{y^2}{4\alpha} + \frac{z^2}{4\alpha} - \alpha e^{-\frac{x}{\alpha}}\right)\frac{\partial}{\partial z},$$

$$\mathbf{X_9} = t\frac{\partial}{\partial x} - \frac{yt}{2\alpha}\frac{\partial}{\partial y} - \frac{zt}{2\alpha}\frac{\partial}{\partial z} - \left(\frac{t^2}{4\alpha} + \frac{y^2}{4\alpha} + \frac{z^2}{4\alpha} + \alpha e^{-\frac{x}{\alpha}}\right)\frac{\partial}{\partial t}.$$

In this case all Noether symmetries are isometries except $\mathbf{Y_0}$, which is given in equation (3.4.1). The first integrals are given in Table 3.14.

Table 3.14: First Integrals

Gen	First Integrals
$\mathbf{X_4}$	$\phi_5 = 2\dot{x} + \frac{e^{\frac{x}{\alpha}}}{\alpha}(t\dot{t} - y\dot{y} - z\dot{z})$
$\mathbf{X_5}$	$\phi_6 = 2e^{\frac{x}{\alpha}}(t\dot{z} - \dot{t}z)$
$\mathbf{X_6}$	$\phi_7 = 2e^{\frac{x}{\alpha}}(t\dot{y} - \dot{t}y)$
$\mathbf{X_7}$	$\phi_8 = 2\dot{x}y + \frac{e^{\frac{x}{\alpha}}}{2\alpha}\left[2yt\dot{t} - 2yz\dot{z} + (z^2 - y^2 - t^2 + 4\alpha^2 e^{\frac{-x}{\alpha}})\dot{y}\right]$
$\mathbf{X_8}$	$\phi_9 = 2\dot{x}z + \frac{e^{\frac{x}{\alpha}}}{2\alpha}\left[2zt\dot{t} - 2yz\dot{y} + (y^2 - z^2 - t^2 + 4\alpha^2 e^{\frac{-x}{\alpha}})\dot{z}\right]$
$\mathbf{X_9}$	$\phi_{10} = 2\dot{x}t + \frac{e^{\frac{x}{\alpha}}}{2\alpha}\left[-2y\dot{y}t - 2z\dot{z}t + (z^2 + y^2 + t^2 + 4\alpha^2 e^{\frac{-x}{\alpha}})\dot{t}\right]$

The Lie algebra of $\mathbf{X_4}$, $\mathbf{X_5}$, $\mathbf{X_6}$, $\mathbf{X_7}$, $\mathbf{X_8}$, and $\mathbf{X_9}$ along with the Noether symmetry generators of equation (3.4.1) is

$[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_0}, \mathbf{X_6}] = \mathbf{X_1}$, $[\mathbf{X_1}, \mathbf{X_6}] = \mathbf{X_0}$, $[\mathbf{X_3}, \mathbf{X_6}] = -\mathbf{X_5}$,
$[\mathbf{X_0}, \mathbf{X_5}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_5}] = \mathbf{X_0}$, $[\mathbf{X_3}, \mathbf{X_5}] = \mathbf{X_6}$, $[\mathbf{X_6}, \mathbf{X_5}] = \mathbf{X_3}$, $[\mathbf{X_0}, \mathbf{X_4}] = \frac{-1}{2\alpha}\mathbf{X_0}$,
$[\mathbf{X_1}, \mathbf{X_4}] = \frac{-1}{2\alpha}\mathbf{X_1}$, $[\mathbf{X_2}, \mathbf{X_4}] = \frac{-1}{2a}\mathbf{X_2}$, $[\mathbf{X_0}, \mathbf{X_7}] = \frac{-1}{2\alpha}\mathbf{X_3}$, $[\mathbf{X_0}, \mathbf{X_4}] = \mathbf{X_4}$,
$[\mathbf{X_2}, \mathbf{X_7}] = \frac{-1}{2\alpha}\mathbf{X_3}$, $[\mathbf{X_3}, \mathbf{X_7}] = -\mathbf{X_8}$, $[\mathbf{X_0}, \mathbf{X_4}] = \frac{-1}{2\alpha}\mathbf{X_0}$, $[\mathbf{X_0}, \mathbf{X_8}] = \frac{-1}{2\alpha}\mathbf{X_5}$,
$[\mathbf{X_1}, \mathbf{X_8}] = \frac{1}{2\alpha}\mathbf{X_3}$, $[\mathbf{X_2}, \mathbf{X_8}] = \mathbf{X_4}$, $[\mathbf{X_0}, \mathbf{X_4}] = -\mathbf{X_8}$, $[\mathbf{X_0}, \mathbf{X_9}] = \mathbf{X_4}$,
$[\mathbf{X_1}, \mathbf{X_9}] = \frac{-1}{2\alpha}\mathbf{X_6}$, $[\mathbf{X_2}, \mathbf{X_9}] = \frac{-1}{2\alpha}\mathbf{X_5}$, $[\mathbf{X_6}, \mathbf{X_7}] = \mathbf{X_9}$, $[\mathbf{X_6}, \mathbf{X_9}] = \mathbf{X_7}$,
$[\mathbf{X_5}, \mathbf{X_8}] = \mathbf{X_9}$, $[\mathbf{X_4}, \mathbf{X_7}] = \frac{-1}{2\alpha}\mathbf{X_7}$, $[\mathbf{X_4}, \mathbf{X_8}] = \frac{-1}{2\alpha}\mathbf{X_8}$, $[\mathbf{X_4}, \mathbf{X_9}] = \frac{-1}{2\alpha}\mathbf{X_9}$,
$[\mathbf{X_i}, \mathbf{X_j}] = 0$ and $[\mathbf{X_i}, \mathbf{Y_0}] = 0$ otherwise.

3.10 Seventeen Noether Symmetries and First Integrals

We get only one solution where 17 Noether symmetries exist:

Solution:

Coefficients of the spacetimes are

$$\nu(x) = \mu(x) = c.$$

Components of the Noether symmetry generators are

$$\begin{aligned}
&\xi = c_1 s^2 + c_2 s + c_3,\\
&\eta^0 = c_4 y + c_5 z + c_6 x + c_{10} s + c_2 \frac{t}{2} + c_1 st + c_{17},\\
&\eta^1 = c_6 t - c_7 y - c_8 z + c_9 + c_2 \frac{x}{2} + c_1 sx - c_{13} s,\\
&\eta^2 = c_4 t + c_7 x + c_2 \frac{y}{2} - c_{11} s + c_1 sy - c_{14} z + c_{15},\\
&\eta^3 = c_5 t + c_8 x + c_2 \frac{z}{2} - c_{12} s + c_1 sz + c_{14} y + c_{16},\\
&A = c_1(t^2 - x^2 - y^2 - z^2) + 2c_{10} t + 2c_{13} x + 2c_{11} y + 2c_{12} z + c_{18}.
\end{aligned}$$

The spacetime in this is the famous Minkowski spacetime

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \tag{3.10.1}$$

The Noether symmetry generators other than the minimal set are

$$\begin{aligned}
&\mathbf{X_4} = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, \quad \mathbf{X_5} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \quad \mathbf{X_6} = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x},\\
&\mathbf{X_7} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad \mathbf{X_8} = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad \mathbf{X_9} = \frac{\partial}{\partial x},\\
&\mathbf{Y_1} = s\frac{\partial}{\partial t}, \quad A_1 = 2t, \quad \mathbf{Y_2} = 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},\\
&\mathbf{Y_3} = s\frac{\partial}{\partial y}, \quad A_3 = -2y, \quad \mathbf{Y_4} = s\frac{\partial}{\partial z}, \quad A_4 = -2z, \mathbf{Y_6} = s\frac{\partial}{\partial x}, \quad A_6 = -2x
\end{aligned}$$

$$\mathbf{Y_5} = s^2\frac{\partial}{\partial s} + st\frac{\partial}{\partial t} + sx\frac{\partial}{\partial x} + sy\frac{\partial}{\partial y} + sz\frac{\partial}{\partial z}, \quad A_5 = t^2 - x^2 - y^2 - z^2. \tag{3.10.2}$$

The symmetries $\mathbf{X_4}$ to $\mathbf{X_9}$ are isometries and $\mathbf{Y_2}$ is homothety. The first integrals are given in Table 3.15.

Table 3.15: First Integrals

Gen	First Integrals
$\mathbf{X_4}$	$\phi_5 = 2(t\dot{y} - \dot{t}y)$
$\mathbf{X_5}$	$\phi_6 = 2(t\dot{z} - \dot{t}z)$
$\mathbf{X_6}$	$\phi_7 = 2(t\dot{x} - \dot{t}x)$
$\mathbf{X_7}$	$\phi_8 = 2(x\dot{y} - \dot{x}y)$
$\mathbf{X_8}$	$\phi_9 = 2(x\dot{z} - \dot{x}z)$
$\mathbf{X_9}$	$\phi_{10} = 2\dot{x}$
$\mathbf{Y_1}$	$\phi_{11} = 2(t - s\dot{t})$
$\mathbf{Y_2}$	$\phi_{12} = 2s\mathcal{L} - 2[t\dot{t} - x\dot{x} - y\dot{y} - z\dot{z}]$
$\mathbf{Y_3}$	$\phi_{13} = 2(s\dot{y} - y)$
$\mathbf{Y_4}$	$\phi_{14} = 2(s\dot{z} - z)$
$\mathbf{Y_5}$	$\phi_{15} = s^2\mathcal{L} - s[t\dot{t} + x\dot{x} + y\dot{y} + z\dot{z}] + t^2 - x^2 - y^2 - z^2$
$\mathbf{Y_6}$	$\phi_{16} = 2(s\dot{x} - x)$

The Lie algebra of $\mathbf{X_4}$, $\mathbf{X_5}$, $\mathbf{X_6}$, $\mathbf{X_7}$, $\mathbf{X_8}$, $\mathbf{X_9}$, $\mathbf{Y_1}$, $\mathbf{Y_2}$, $\mathbf{Y_3}$, $\mathbf{Y_4}$, $\mathbf{Y_5}$ and $\mathbf{Y_6}$ along with the symmetry generators of equation (3.4.1) is
$[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_0}, \mathbf{X_4}] = \mathbf{X_1}$, $[\mathbf{X_1}, \mathbf{X_4}] = \mathbf{X_0}$, $[\mathbf{X_3}, \mathbf{X_4}] = -\mathbf{X_5}$,
$[\mathbf{X_0}, \mathbf{X_5}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_5}] = \mathbf{X_0}$, $[\mathbf{X_3}, \mathbf{X_5}] = \mathbf{X_4}$, $[\mathbf{X_4}, \mathbf{X_5}] = \mathbf{X_3}$, $[\mathbf{X_0}, \mathbf{X_6}] = \mathbf{X_9}$,
$[\mathbf{X_1}, \mathbf{X_7}] = -\mathbf{X_9}$, $[\mathbf{X_3}, \mathbf{X_7}] = \mathbf{X_8}$, $[\mathbf{X_2}, \mathbf{X_8}] = -\mathbf{X_9}$, $[\mathbf{X_3}, \mathbf{X_8}] = \mathbf{X_7}$, $[\mathbf{Y_0}, \mathbf{Y_3}] = -\mathbf{X_1}$,
$[\mathbf{Y_0}, \mathbf{Y_4}] = -\mathbf{X_2}$, $[\mathbf{Y_0}, \mathbf{Y_1}] = \mathbf{X_0}$, $[\mathbf{X_0}, \mathbf{Y_2}] = \mathbf{X_0}$, $[\mathbf{X_1}, \mathbf{Y_2}] = \mathbf{X_1}$, $[\mathbf{Y_0}, \mathbf{Y_6}] = -\mathbf{X}_0$,
$[\mathbf{X_0}, \mathbf{Y_5}] = \mathbf{Y_1}$, $[\mathbf{X_1}, \mathbf{Y_5}] = \mathbf{Y_3}$, $[\mathbf{X_2}, \mathbf{Y_5}] = \mathbf{Y_4}$, $[\mathbf{Y_0}, \mathbf{Y_5}] = \mathbf{Y_2}$, $[\mathbf{X_2}, \mathbf{Y_2}] = \mathbf{X_2}$,
$[\mathbf{Y_0}, \mathbf{Y_2}] = 2\mathbf{Y_0}$, $[\mathbf{X_4}, \mathbf{X_6}] = -\mathbf{X_7}$, $[\mathbf{X_4}, \mathbf{X_7}] = -\mathbf{X_6}$, $[\mathbf{X_5}, \mathbf{X_6}] = -\mathbf{X_8}$, $[\mathbf{X_5}, \mathbf{X_8}] = -\mathbf{X_6}$,
$[\mathbf{X_5}, \mathbf{Y_4}] = -\mathbf{Y_1}$, $[\mathbf{X_5}, \mathbf{Y_1}] = \mathbf{Y_4}$, $[\mathbf{X_6}, \mathbf{X_7}] = \mathbf{X_4}$, $[\mathbf{X_6}, \mathbf{X_8}] = \mathbf{X_5}$, $[\mathbf{X_6}, \mathbf{X_9}] = -\mathbf{X_0}$,
$[\mathbf{X_6}, \mathbf{Y_1}] = -\mathbf{Y_6}$, $[\mathbf{X_6}, \mathbf{Y_6}] = \mathbf{Y_1}$, $[\mathbf{X_7}, \mathbf{X_8}] = -\mathbf{X_3}$, $[\mathbf{X_7}, \mathbf{X_9}] = \mathbf{X_2}$, $[\mathbf{X_7}, \mathbf{Y_3}] = \mathbf{X_9}$,
$[\mathbf{X_7}, \mathbf{Y_6}] = -\mathbf{Y_3}$, $[\mathbf{X_8}, \mathbf{X_9}] = -\mathbf{X_2}$, $[\mathbf{X_8}, \mathbf{Y_4}] = \mathbf{Y_6}$, $[\mathbf{X_8}, \mathbf{Y_6}] = -\mathbf{Y_4}$, $[\mathbf{X_9}, \mathbf{Y_5}] = -\mathbf{Y_6}$,
$[\mathbf{X_9}, \mathbf{Y_2}] = \mathbf{X_9}$, $[\mathbf{Y_3}, \mathbf{Y_2}] = -\mathbf{Y_3}$, $[\mathbf{Y_4}, \mathbf{Y_2}] = -\mathbf{Y_4}$, $[\mathbf{Y_1}, \mathbf{Y_2}] = -\mathbf{Y_1}$, $[\mathbf{Y_2}, \mathbf{Y_5}] = \mathbf{Y_5}$,
$[\mathbf{X_i}, \mathbf{X_j}] = 0$, $[\mathbf{X_i}, \mathbf{Y_j}] = 0$ and $[\mathbf{Y_i}, \mathbf{Y_j}] = 0$ otherwise.

Chapter 4

Approximate Noether Symmetries of Arc Length Minimizing Lagrangian of Time Conformal Plane Symmetric Spacetimes

4.1 Introduction

Energy and momentum are important quantities whose definitions have been a focus of many investigations in general relativity. Although several attempts are made to define them [40, 41, 65], unfortunately, there are no accepted definitions so for.

In 1915, Emmy Noether [58] proved that there is one to one correspondence between the symmetries of the action (Noether symmetries) and conservation laws [6, 48, 54, 57]. That is for every Noether symmetry, there exists a conservation law (conserved quantity). Here we find the approximate Noether symmetries of the action of time conformal plane symmetric spacetime for the investigation of energy and momentum [37, 47, 55, 71, 79]. Fortunately, we find three different approximate Noether symmetries, one corresponds to the energy, the second one corresponds to scaling and the third one corresponds to the Lorentz transformation.

For this purpose we take first order perturbed plane symmetric metric, find its arc length minimizing Lagrangian density and use it in the approximate Noether symmetry equation. We obtain a system of 19 PDEs. The solutions of this system give us all those spacetimes which admit approximate Noether symmetry/symmetries. These spacetimes are not the exact gravitational wave spacetimes, but will help us understanding these spacetimes [24,77,78,80]. The approximate Noether symmetries correspond to the approximate first integrals which define the conservation laws in the respective spacetimes [74–76]. These are the approximate gravitational wave spacetimes which provide us the insight and information about the exact gravitational wave spacetimes [26–30, 39]. We present here only those cases where the approximate Noether symmetry(ies) exist(s).

4.1.1 Perturbed Plane Symmetric Spacetime and its Lagrangian

We take the plane symmetric static spacetime as

$$ds_e^2 = e^{\nu(x)}dt^2 - dx^2 - e^{\mu(x)}(dy^2 + dz^2), \tag{4.1.1}$$

and the corresponding Lagrangian density is

$$\mathcal{L}_e = e^{\nu(x)}\dot{t}^2 - \dot{x}^2 - e^{\mu(x)}(\dot{y}^2 + \dot{z}^2), \tag{4.1.2}$$

where " $\cdot$ " denotes differentiation with respect to s. We perturb the metric given by equation (4.1.1) by using the general time conformal factor $e^{\epsilon f(t)}$, which gives

$$ds^2 = e^{\epsilon f(t)}ds_e^2, \tag{4.1.3}$$

and the corresponding perturbed Lagrangian density takes the form

$$\mathcal{L} = e^{\epsilon f(t)}\mathcal{L}_e. \tag{4.1.4}$$

The first order perturbed plane symmetric metric and its Lagrangian density, in expanded form are

$$\begin{aligned} ds^2 &= e^{\nu(x)}dt^2 - dx^2 - e^{\mu(x)}(dy^2 + dz^2) + \epsilon f(t)\{e^{\nu(x)}dt^2 - dx^2 - e^{\mu(x)}(dy^2 + dz^2)\}, \\ \mathcal{L} &= e^{\nu(x)}\dot{t}^2 - \dot{x}^2 - e^{\mu(x)}(\dot{y}^2 + \dot{z}^2) + \epsilon f(t)\{e^{\nu(x)}\dot{t}^2 - \dot{x}^2 - e^{\mu(x)}(\dot{y}^2 + \dot{z}^2)\}. \end{aligned} \tag{4.1.5}$$

4.1.2 First Order Approximate Noether Symmetry and Noether Symmetry Equation

The operator

$$\mathbf{X}_e = \xi_e \frac{\partial}{\partial s} + \eta^i_e \frac{\partial}{\partial x^i}, \tag{4.1.6}$$

is the Noether symmetry generator if it satisfies equation (2.2.41) for first order extension

$$\mathbf{X}^{[1]}_e = \mathbf{X}_e + \eta^i_{es} \frac{\partial}{\partial \dot{x}^i}, \tag{4.1.7}$$

first order Lagrangian density given in equation (4.1.2) and differential operator (3.2.4), where the subscript 'e' denotes the exactness of the generator and x^i refer to the dependent variables t, x, y, z.

The first order approximate Noether symmetry is defined as

$$\mathbf{X} = \mathbf{X}_e + \epsilon \mathbf{X}_a, \tag{4.1.8}$$

up to the gauge $A = A_e + \epsilon A_a$, where

$$\mathbf{X}_a = \xi_a \frac{\partial}{\partial s} + \eta^i_a \frac{\partial}{\partial x^i}, \tag{4.1.9}$$

is the approximate Noether symmetry and A_a is the approximate part of the gauge function.

$\mathbf{X}$ is the first order approximate Noether symmetry if it satisfies the equation

$$\mathbf{X}^{[1]} L + (D\xi) L = DA, \tag{4.1.10}$$

where $\mathbf{X}^{[1]}$ is the first order prolongation of the first order approximate Noether symmetry $\mathbf{X}$ given in equation (4.1.8). Due to the first order approximation, the equation (4.1.10) splits into two parts as follows

$$\mathbf{X}^{[1]}_e L_e + (D\xi_e) L_e = DA_e, \tag{4.1.11}$$

$$\mathbf{X}^{[1]}_a L_e + \mathbf{X}^{[1]}_e L_a + (D\xi_e) L_a + (D\xi_a) L_e = DA_a. \tag{4.1.12}$$

All η^i_e, η^i_a, ξ_e, ξ_a, A_e and A_a are the functions of s, t, x, y, z and η^i_{es}, η^i_{as} are functions of s, t, x, y, z, $\dot{t}$, $\dot{x}$, $\dot{y}$, $\dot{z}$. Equation (4.1.11) is the exact Noether symmetry equation the solution of which is given in Chapter 3. For the approximate Noether symmetries we will solve equation (4.1.12) which provide a system of 19 PDEs whose solution provides us the cases where the approximate Noether symmetry(ies) exist(s).

4.1.3 Determining PDEs for Approximate Noether Symmetries

From equation (4.1.12) we obtain the following system of 19 PDEs.

$$\begin{aligned}
&\xi_{a,t}=0, \quad \xi_{a,x}=0, \xi_{a,y}=0, \quad \xi_{a,z}=0, \quad A_{a,s}=0,\\
&2(\eta^0_{a,s}+f(t)\eta^0_{e,s})e^{\nu(x)}-A_{a,t}=0, \quad 2(\eta^1_{a,s}+f(t)\eta^1_{e,s})+A_{a,x}=0,\\
&f_t(t)\eta^0_e+(\eta^1_a+f(t)\eta^1 e)\nu^{'}(x)+2(\eta^0_{a,t}+f(t)\eta^0_{e,t})-f(t)\xi_{e,s}-\xi_{a,s}=0,\\
&f_t(t)\eta^0_e+(\eta^1_a+f(t)\eta^1_e)\mu^{'}(x)+2(\eta^2_{a,y}+f(t)\eta^2_{e,y})-f(t)\xi_{e,s}-\xi_{a,s}=0,\\
&f_t(t)\eta^0_e+(\eta^1_a+f(t)\eta^1_e)\mu^{'}(x)+2(\eta^3_{a,z}+f(t)\eta^3_{e,z})-f(t)\xi_{e,s}-\xi_{a,s}=0,\\
&f_t(t)\eta^0_e+2\eta^1_{a,x}+2f(t)\eta^1_{e,x}-f(t)\xi_{e,s}-\xi_{a,s}=0, \qquad (4.1.13)\\
&2(\eta^2_{a,s}+f(t)\eta^2_{e,s})e^{\mu(x)}+A_{a,y}=0, 2(\eta^3_{a,s}+f(t)\eta^3_{e,s})e^{\mu(x)}+A_{a,z}=0,\\
&\eta^1_{a,y}+f(t)\eta^1_{e,y}+(\eta^2_{a,x}+f(t)\eta^2_{e,x})e^{\mu(x)}=0, \quad \eta^1_{a,z}+f(t)\eta^1_{e,z}+(\eta^3_{a,x}+f(t)\eta^3_{e,x})e^{\mu(x)}=0,\\
&\eta^1_{a,t}+f(t)\eta^1_{e,t}-(\eta^0_{a,x}+f(t)\eta^0_{e,x})e^{\nu(x)}=0, e^{\nu(x)}(f(t)\eta^0_{e,y}+\eta^0_{a,y})-e^{\mu(x)}(f(t)\eta^2_{e,t}+\eta^2_{a,t})=0,\\
&e^{\nu(x)}(f(t)\eta^0_{e,z}+\eta^0_{a,z})-e^{\mu(x)}(f(t)\eta^3_{e,t}+\eta^3_{a,t})=0, \quad (f(t)\eta^2_{e,z}+\eta^2_{a,z})+(f(t)\eta^3_{e,y}+\eta^3_{a,y})=0.
\end{aligned}$$

Solutions of this system provide those spacetimes where the approximate Noether symmetries exist.

4.2 Solutions of the Perturbed System Given in Equations (4.1.13)

4.2.1 Five Noether Symmetries and Time Conformal Spacetime

There are infinitely many plane symmetric metrics, the actions of the Lagrangian of which admit only five (minimal set of) Noether symmetries. We list few of them in Table 4.1. The exact Noether symmetry generators in these cases are

$$\mathbf{Y}_0=\frac{\partial}{\partial s}, \quad \mathbf{X}_1=\frac{\partial}{\partial y}, \quad \mathbf{X}_2=\frac{\partial}{\partial z}, \quad \mathbf{X}_3=y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}, \qquad (4.2.1)$$

and the approximate Noether symmetry is

$$\mathbf{X_0}=\frac{\partial}{\partial t}+\frac{\epsilon s}{\alpha}\frac{\partial}{\partial s}. \qquad (4.2.2)$$

The approximate Noether symmetry generator given in equation (4.2.2) corresponds to the energy in the given spacetime. The first integral corresponding to $\mathbf{X}_0$ is

$$\phi_0 = -2e^{\nu(x)}\dot{t} - \frac{\epsilon}{\alpha}(2t\dot{t}e^{\nu(x)} - \mathcal{L}s).$$

Table 4.1: Metrics for five symmetries

No.	$\nu(x)$	$\mu(x)$	$f(t)$
1.	$2\ln(\frac{x}{\alpha})$	$\frac{x}{\alpha}$	$\frac{t}{\alpha}$
2.	$\frac{x}{\alpha}$	$2\ln(\frac{x}{\alpha})$	$\frac{t}{\alpha}$
3.	$2\ln(\frac{x}{\alpha})$	$2\ln\cosh(\frac{x}{\alpha})$	$\frac{t}{\alpha}$
4.	$2\ln(\frac{x}{\alpha})$	$2\ln\cos(\frac{x}{\alpha})$	$\frac{t}{\alpha}$
5.	$\frac{x}{\alpha}$	$2\ln\cosh(\frac{x}{\alpha})$	$\frac{t}{\alpha}$
6.	$\frac{x}{\alpha}$	$2\ln\cos(\frac{x}{\alpha})$	$\frac{t}{\alpha}$
7.	$\frac{x}{\alpha}$	$non-linear$	$\frac{t}{\alpha}$
8.	$a\ln(\frac{x}{\alpha})$	$\mu''(x) \neq 0, \mu(x) \neq a\ln\frac{x}{\alpha}$	$\frac{t}{\alpha}$
9.	$\nu(x) \neq \mu(x), \nu'(x) \neq 0$	$\mu''(x) \neq 0, \mu(x) \neq a\ln\frac{x}{\alpha}$	$\frac{t}{\alpha}$
10.	$\nu''(x) \neq 0, \nu(x) \neq a\ln\frac{x}{\alpha}$	$\nu(x) \neq \mu(x), \mu'(x) \neq 0$	$\frac{t}{\alpha}$

All the ten classes given in Table 4.1 have five Noether symmetry generators in which only one symmetry generator $\mathbf{X_0}$ has approximate part.

4.2.2 Six Noether Symmetries and Time Conformal Spacetimes

There are two classes of six Noether symmetries, where the approximate part(s) exist(s). The action of the Lagrangians of the metrics given in this section admit six Noether symmetries in which some of the symmetries admit approximation. The detail calculation is given below:

Solution-I:

Using the exact solution

$$\eta_e^0 = c_3, \quad \eta_e^1 = \frac{c_1 x}{2}, \quad \eta_e^2 = \frac{c_1(2-k)y}{4} - c_4 z + c_5, \quad \eta_e^3 = \frac{c_1(2-k)z}{4} - c_4 z + c_6,$$
$$\xi_e^0 = c_1 s + c_0, \quad A = c, \quad \nu(x) = 2\ln\left(\frac{x}{\alpha}\right), \quad \mu(x) = k\ln\left(\frac{x}{\alpha}\right),$$

in system (4.1.13) we obtain the approximate system as

$$
\begin{aligned}
&\xi^0_{a,t}=0, \quad \xi^0_{a,x}=0, \quad \xi^0_{a,y}=0, \quad \xi^0_{a,z}=0, \quad A_{a,s}=0,\\
&A_{a,t}-2\frac{x^2}{\alpha^2}\eta^0_{a,s}=0, \quad A_{a,x}+2\eta^1_{a,s}=0,\\
&A_{a,y}+2\frac{x^k}{\alpha^k}\eta^2_{a,s}=0, \quad A_{a,z}+2\frac{x^k}{\alpha^k}\eta^3_{a,s}=0,\\
&\eta^1_{a,t}-\frac{x^2}{\alpha^2}\eta^0_{a,x}=0, \quad \eta^1_{a,y}+\frac{x^k}{\alpha^k}\eta^2_{a,x}=0,\\
&\eta^1_{a,z}+\frac{x^k}{\alpha^k}\eta^3_{a,x}=0, \quad \eta^2_{a,z}+\eta^3_{a,y}=0,\\
&\frac{x^2}{\alpha^2}\eta^0_{a,y}-\frac{x^k}{\alpha^k}\eta^2_{a,t}=0, \quad \frac{x^2}{\alpha^2}\eta^0_{a,z}-\frac{x^k}{\alpha^k}\eta^3_{a,t}=0,\\
&c_3 f_t(t)+2\eta^1_{a,x}-\xi^0_{a,s}=0, \quad c_3 f_t(t)+\frac{2}{x}\eta^1_a+2\eta^0_{a,t}-\xi^0_{a,s}=0,\\
&c_3 f_t(t)+\frac{k}{x}\eta^1_a+2\eta^2_{a,y}-\xi^0_{a,s}=0, \quad c_3 f_t(t)+\frac{k}{x}\eta^1_a+2\eta^3_{a,z}-\xi^0_{a,s}=0.
\end{aligned} \tag{4.2.3}
$$

Solving the above system the components of the Noether symmetry generators and the value of $f(t)$ become

$$
\begin{aligned}
\eta^0_a &= \frac{-c_3 t}{2\alpha}+\frac{b_1 t}{2}, \quad \eta^1_a=\frac{b_1 x}{2}-\frac{c_3 x}{2\alpha},\\
\eta^2_a &= \frac{c_3(k-2)y}{4\alpha}+\frac{b_1(2-k)y}{4}-b_2 z+b_3,\\
\eta^3_a &= \frac{c_3(k-2)z}{4\alpha}+\frac{b_1(2-k)z}{4}+b_2 y+b_4,\\
A_a &= b, \quad \xi^0_a=b_1 s+b_0, \quad f(t)=\frac{t}{\alpha}.
\end{aligned}
$$

Metric of the spacetime takes the form

$$
\begin{aligned}
ds^2 &= \left(\frac{x}{\alpha}\right)^2 dt^2-dx^2-\left(\frac{x}{\alpha}\right)^k(dy^2+dz^2)+\\
&\frac{\epsilon t}{\alpha}\left(\left(\frac{x}{\alpha}\right)^2 dt^2-dx^2-\left(\frac{x}{\alpha}\right)^k(dy^2+dz^2)\right), \quad k\neq 0,2, \quad \alpha\neq 0.
\end{aligned} \tag{4.2.4}
$$

The approximate Noether symmetry generator in this case is

$$
\mathbf{X}_0=\frac{\partial}{\partial t}-\frac{\epsilon}{4\alpha}\left(2t\frac{\partial}{\partial t}+2x\frac{\partial}{\partial x}+(2-k)y\frac{\partial}{\partial y}+(2-k)y\frac{\partial}{\partial z}\right). \tag{4.2.5}
$$

The conservation law or first integral corresponding to the Noether symmetry generator given in equation (4.2.5) is

$$
\phi_0=2\left[-\left(\frac{x}{\alpha}\right)^2\dot{t}+\frac{\epsilon}{\alpha}\left(\frac{t\dot{t}}{2}\left(\frac{x}{\alpha}\right)^2+\frac{x\dot{x}}{2}+(2-k)\frac{y\dot{y}}{4}\left(\frac{x}{\alpha}\right)^k+(2-k)\frac{z\dot{z}}{4}\left(\frac{x}{\alpha}\right)^k\right)\right].
$$

Solution-II:

By substituting the exact solution

$$\eta_e^0 = c_1\frac{2-k}{4}y + c_2, \quad \eta_e^1 = \frac{c_1 x}{2}, \quad \eta_e^2 = -c_3 z + c_4, \quad \eta_e^3 = c_3 z + c_5,$$
$$\xi_e^0 = c_1 s + c_0, \quad A = c, \quad \nu(x) = k\ln\left(\frac{x}{\alpha}\right), \quad \mu(x) = 2\ln\left(\frac{x}{\alpha}\right),$$

in system given in equations (4.1.13) we get

$$\begin{aligned}
&\xi^0_{a,t} = 0, \quad \xi^0_{a,x} = 0, \quad \xi^0_{a,y} = 0, \quad \xi^0_{a,z} = 0, \quad A_{a,s} = 0,\\
&A_{a,t} - 2\frac{x^k}{\alpha^k}\eta^0_{a,s} = 0, \quad A_{a,x} + 2\eta^1_{a,s} = 0,\\
&A_{a,y} + 2\frac{x^2}{\alpha^2}\eta^2_{a,s} = 0, \quad A_{a,z} + 2\frac{x^2}{\alpha^2}\eta^3_{a,s} = 0,\\
&\eta^1_{a,t} - \frac{x^k}{\alpha^k}\eta^0_{a,x} = 0, \quad \eta^1_{a,y} + \frac{x^2}{\alpha^2}\eta^2_{a,x} = 0,\\
&\eta^1_{a,z} + \frac{x^2}{\alpha^2}\eta^3_{a,x} = 0, \quad \eta^2_{a,z} + \eta^3_{a,y} = 0,\\
&\frac{x^2}{\alpha^2}\eta^0_{a,y} - \frac{x^2}{\alpha^2}\eta^2_{a,t} = 0, \quad \frac{x^2}{\alpha^2}\eta^0_{a,z} - \frac{x^2}{\alpha^2}\eta^3_{a,t} = 0,\\
&\left(\frac{c_1(2-k)t}{4} + c_2\right)f_t(t) + 2\eta^1_{a,x} - \xi^0_{a,s} = 0,\\
&\left(\frac{c_1(2-k)t}{4} + c_2\right)f_t(t) + \frac{2}{x}\eta^1_a + 2\eta^0_{a,t} - \xi^0_{a,s} = 0,\\
&\left(\frac{c_1(2-k)t}{4} + c_2\right)f_t(t) + \frac{k}{x}\eta^1_a + 2\eta^2_{a,y} - \xi^0_{a,s} = 0,\\
&\left(\frac{c_1(2-k)t}{4} + c_2\right)f_t(t) + \frac{k}{x}\eta^1_a + 2\eta^3_{a,z} - \xi^0_{a,s} = 0.
\end{aligned} \tag{4.2.6}$$

Solving this system we obtain the following solution

$$\begin{aligned}
&\eta^0_a = -\frac{c_1\alpha^2}{8\alpha x^{k-2}} - \frac{c_1 t^2(2-k)^2}{32\alpha} - \frac{c_2 t(2-k)}{4\alpha} + \frac{b_1 t(2-k)}{4} + b_4,\\
&\eta^1_a = \frac{b_2 x s}{2} - \frac{c_1(2-k)tx}{8\alpha} - \frac{c_2 x}{2\alpha} + \frac{b_1 x}{2},\\
&\eta^2_a = -b_4 z + b_5, \quad \eta^3_a = b_4 y + b_6, \quad A_a = b_3,\\
&\xi^0_a = b_1 s + b_0, \quad f(t) = \frac{t}{\alpha}.
\end{aligned} \tag{4.2.7}$$

The spacetime takes the form

$$\begin{aligned}
&ds^2 = \left(\frac{x}{\alpha}\right)^k dt^2 - dx^2 - \left(\frac{x}{\alpha}\right)^2(dy^2+dz^2) +\\
&\frac{\epsilon t}{\alpha}\Bigg(\left(\frac{x}{\alpha}\right)^k dt^2 - dx^2 - \left(\frac{x}{\alpha}\right)^2(dy^2+dz^2)\Bigg), \quad k \neq 0, 2, \quad \alpha \neq 0.
\end{aligned} \tag{4.2.8}$$

The following two approximate Noether symmetry generators are obtained

$$\mathbf{X_0} = \frac{\partial}{\partial t} - \frac{\epsilon}{4\alpha}\left(t(2-k)\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x}\right), \tag{4.2.9}$$
$$\mathbf{X_1} = s\frac{\partial}{\partial s} + \frac{t(2-k)}{4}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} - \epsilon\left((\frac{x^2\alpha^k}{8\alpha x^k} - \frac{t^2(2-k)^2}{32\alpha})\frac{\partial}{\partial t} + \frac{(2-k)tx}{8\alpha}\frac{\partial}{\partial x}\right). \tag{4.2.10}$$

The first integrals corresponding to $\mathbf{X}_0$ and $\mathbf{X}_1$ are given in the following table

Table 4.2: First integrals

Gen	First integrals
$\mathbf{X_0}$	$\phi_0 = 2\left[-\left(\frac{x}{\alpha}\right)^k \dot{t} - \frac{\epsilon}{\alpha}\{\left(\frac{x}{\alpha}\right)^k \left(t\dot{t} - \frac{(2-k)t\dot{t}}{4}\right) + \frac{x\dot{x}}{2}\}\right]$
$\mathbf{X_1}$	$\phi_1 = s\mathcal{L} - \left(\frac{x}{\alpha}\right)^k \frac{t\dot{t}(2-k)}{2} + x\dot{x}$
	$-\frac{\epsilon}{\alpha}\left[-\frac{x^2\dot{t}}{4} + \left(\frac{x}{\alpha}\right)^k \left(\frac{t^2\dot{t}(2-k)^2}{16} + \frac{t^2\dot{t}(2-k)}{2}\right) - \frac{xt\dot{t}(2+k)}{4}\right]$

4.2.3 Eight Noether Symmetries and Time Conformal Spacetime

We have only one spacetime, the action of the arc length minimizing Lagrangian of which admits eight Noether symmetries in which four symmetries admit approximate parts:

Solution

Putting the exact solution

$$\begin{aligned}
&\eta^0 = \frac{c_1(2-k)t}{4} + c_2y + c_3z + c_4, \quad \eta^1 = \frac{c_1x}{2},\\
&\eta^2 = \frac{c_1(2-k)y}{4} + c_2t - c_5z + c_6, \quad \eta^3 = \frac{c_1(2-k)z}{4} + c_3t + c_5y + c_7,\\
&A = c, \quad \xi^0 = c_1s + c_0, \quad \nu(x) = k\ln\left(\frac{x}{\alpha}\right) = \mu(x),
\end{aligned} \tag{4.2.11}$$

in system given in equations (4.1.13) we get the approximate system of 19 PDEs as

$$\begin{aligned}
&\xi^0_{a,t} = 0, \quad \xi^0_{a,x} = 0, \quad \xi^0_{a,y} = 0, \quad \xi^0_{a,z} = 0, \quad A_{a,s} = 0,\\
&A_{a,t} - 2\frac{x^k}{\alpha^k}\eta^0_{a,s} = 0, \quad A_{a,x} + 2\eta^1_{a,s} = 0,\\
&A_{a,y} + 2\frac{x^k}{\alpha^k}\eta^2_{a,s} = 0, \quad A_{a,z} + 2\frac{x^k}{\alpha^k}\eta^3_{a,s} = 0,\\
&\eta^1_{a,t} - \frac{x^k}{\alpha^k}\eta^0_{a,x} = 0, \quad \eta^1_{a,y} + \frac{x^k}{\alpha^k}\eta^2_{a,x} = 0,\\
&\eta^1_{a,z} + \frac{x^k}{\alpha^k}\eta^3_{a,x} = 0, \quad \eta^2_{a,z} + \eta^3_{a,y} = 0,\\
&\eta^0_{a,y} - \eta^2_{a,t} = 0, \quad \eta^0_{a,z} - \eta^3_{a,t} = 0,
\end{aligned} \tag{4.2.12}$$

$$\begin{aligned}
&\left(\frac{c_1(2-k)t}{4}+c_2y+c_3z+c_4\right)f_t(t)+2\eta^1_{a,x}-\xi^0_{a,s}=0,\\
&\left(\frac{c_1(2-k)t}{4}+c_2y+c_3z+c_4\right)f_t(t)+\frac{k}{x}\eta^1_a+2\eta^0_{a,t}-\xi^0_{a,s}=0,\\
&\left(\frac{c_1(2-k)t}{4}+c_2y+c_3z+c_4\right)f_t(t)+\frac{k}{x}\eta^1_a+2\eta^2_{a,y}-\xi^0_{a,s}=0,\\
&\left(\frac{c_1(2-k)t}{4}+c_2y+c_3z+c_4\right)f_t(t)+\frac{k}{x}\eta^1_a+2\eta^3_{a,z}-\xi^0_{a,s}=0.
\end{aligned} \tag{4.2.13}$$

The solution of the system given in equations (4.2.12) and (4.2.13) takes the form

$$\begin{aligned}
&\eta^0_a=\frac{c_1\alpha^k x^{2-k}}{8\alpha(2-k)}+-\frac{c_1t^2(2-k)^2}{32\alpha}-\frac{c_2ty(2-k)}{4\alpha}-\frac{c_3tz(2-k)}{4\alpha}-\frac{c_4t(2-k)}{4\alpha}+\\
&\frac{b_1t(2-k)}{4}-\frac{c_1y^2(2-k)^2}{32\alpha}+b_6y-\frac{c_1z^2(2-k)^2}{32\alpha}+b_7z+b_8,\\
&\eta^1_a=-\frac{c_1tx(2-k)}{8\alpha}-\frac{c_2xy}{2\alpha}-\frac{c_3xz}{2\alpha}-\frac{c_4x}{2\alpha}+\frac{b_1x}{2},\\
&\eta^2_a=\frac{c_2\alpha^k x^{2-k}}{2\alpha(2-k)}-\frac{c_1ty(2-k)^2}{16\alpha}-\frac{c_2y^2(2-k)}{8\alpha}-\frac{c_3yz(2-k)}{4\alpha}-\frac{c_4y(2-k)}{4\alpha}+\\
&\frac{b_1y(2-k)}{4}-\frac{c_2t^2(2-k)}{8\alpha}+b_6t+\frac{c_2z^2(2-k)}{8\alpha}-b_9z+b_{11},\\
&\eta^3_a=\frac{c_3\alpha^k x^{2-k}}{2\alpha(2-k)}-\frac{c_1tz(2-k)^2}{16\alpha}-\frac{c_2yz(2-k)}{8\alpha}-\frac{c_3z^2(2-k)}{4\alpha}-\frac{c_4z(2-k)}{4\alpha}+\\
&\frac{b_1z(2-k)}{4}-\frac{c_3t^2(2-k)}{8\alpha}+b_7t+\frac{c_3y^2(2-k)}{8\alpha}+b_9y+b_{10},\\
&A_a=b_4,\quad \xi^0_a=b_1s+b_0,\quad f(t)=\frac{t}{\alpha}.
\end{aligned} \tag{4.2.14}$$

The metric in this case is

$$\begin{aligned}
&ds^2=\left(\frac{x}{\alpha}\right)^k(dt^2-dy^2-dz^2)-dx^2+\\
&\frac{\epsilon t}{\alpha}\left(\left(\frac{x}{\alpha}\right)^k(dt^2-dy^2-dz^2)-dx^2\right),k\neq 0,2.
\end{aligned} \tag{4.2.15}$$

The Noether symmetry generators which have the approximate parts are

$$\begin{aligned}
&\mathbf{X_0}=\frac{\partial}{\partial t}-\frac{\epsilon}{4\alpha}\left((2-k)t\frac{\partial}{\partial t}+2x\frac{\partial}{\partial x}+(2-k)y\frac{\partial}{\partial y}+(2-k)z\frac{\partial}{\partial z}\right),\\
&\mathbf{X_1}=s\frac{\partial}{\partial s}+\frac{t(2-k)}{4}\frac{\partial}{\partial t}+\frac{x}{2}\frac{\partial}{\partial x}+\frac{y(2-k)}{4}\frac{\partial}{\partial y}+\frac{z(2-k)}{4}\frac{\partial}{\partial z}-\\
&\frac{\epsilon}{8\alpha}\left(\{x^2(\frac{\alpha^k}{x^k})+\frac{(2-k)^2(t^2+y^2+z^2)}{4}\}\frac{\partial}{\partial t}+tx(2-k)\frac{\partial}{\partial x}+\right.\\
&\left.\frac{ty(2-k)^2}{2}\frac{\partial}{\partial y}+\frac{tz(2-k)^2}{2}\frac{\partial}{\partial z}\right),
\end{aligned}$$

$$\mathbf{X_2} = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y} -$$
$$\frac{\epsilon}{2\alpha}\Big(\frac{ty(2-k)}{2}\frac{\partial}{\partial t} + xy\frac{\partial}{\partial x} + \{\frac{-x^2(\frac{\alpha^k}{x^k})}{2-k} + \frac{(2-k)(t^2+y^2-z^2)}{4}\}\frac{\partial}{\partial y} + \frac{yz(2-k)}{2}\frac{\partial}{\partial z}\Big),$$
$$\mathbf{X_3} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z} -$$
$$\frac{\epsilon}{2\alpha}\Big(\frac{tz(2-k)}{2}\frac{\partial}{\partial t} + xz\frac{\partial}{\partial x} + \frac{yz(2-k)}{2}\frac{\partial}{\partial y} + \{\frac{-x^2(\frac{\alpha^k}{x^k})}{2-k} + \frac{(2-k)(t^2-y^2+z^2)}{4}\}\frac{\partial}{\partial z}\Big).$$

The corresponding first integrals are given in the following table

Table 4.3: First integrals

Gen	First integrals
$\mathbf{X_0}$	$\phi_0 = 2\left[-\left(\frac{x}{\alpha}\right)^k\dot{t} - \frac{\epsilon}{\alpha}\left(\left(\frac{x}{\alpha}\right)^k\left(t\dot{t} - \frac{(2-k)t\dot{t}}{4}\right) + \frac{x\dot{x}}{2} + \left(\frac{x}{\alpha}\right)^k\left(\frac{(2-k)y\dot{y}}{4} + \frac{(2-k)z\dot{z}}{4}\right)\right)\right]$
$\mathbf{X_1}$	$\phi_1 = s\mathcal{L} - \left[\frac{t\dot{t}(2-k)}{2}\left(\frac{x}{\alpha}\right)^k - x\dot{x} - \frac{y\dot{y}(2-k)}{2}\left(\frac{x}{\alpha}\right)^k\frac{z\dot{z}(2-k)}{2}\left(\frac{x}{\alpha}\right)^k\right] - \frac{\epsilon}{\alpha}\Big[-\frac{x^2\dot{t}}{4} + \frac{tx\dot{x}(2-k)}{4} - tx\dot{x} +$ $\left(\frac{x}{\alpha}\right)^k\left(-\frac{\dot{t}(2-k)^2(t^2+y^2+z^2)}{16} + \frac{ty\dot{y}(2-k)^2}{8} + \frac{tz\dot{z}(2-k)^2}{8} + \frac{t^2\dot{t}(2-k)}{2} - \frac{t(2-k)(y\dot{y}+z\dot{z})}{2}\right)\Big]$
$\mathbf{X_2}$	$\phi_2 = 2\left(\frac{x}{\alpha}\right)^k(\dot{y}t - y\dot{t}) - \frac{\epsilon}{\alpha}\left[\left(\frac{x}{\alpha}\right)^k\left(-\frac{ty\dot{t}(2-k)}{2} + \frac{\dot{y}(2-k)(t^2+y^2-z^2)}{4} + 2(ty\dot{t} - t^2\dot{y})\right) + x\dot{x}y - \frac{x^2\dot{y}}{2-k}\right]$
$\mathbf{X_3}$	$\phi_3 = 2\left(\frac{x}{\alpha}\right)^k(\dot{z}t - z\dot{t}) - \frac{\epsilon}{\alpha}\left[\left(\frac{x}{\alpha}\right)^k\left(-\frac{tz\dot{t}(2-k)}{2} + \frac{\dot{z}(2-k)(t^2+z^2-y^2)}{4} + 2(tz\dot{t} - t^2\dot{z})\right) + x\dot{x}z - \frac{x^2\dot{z}}{2-k}\right]$

4.2.4 Nine Noether Symmetries and Time Conformal Spacetime

We have only one class of nine Noether symmetries where the approximate symmetries exist:

Solution:

Substituting the exact solution

$$\eta^0 = c_3y + c_4z + c_5, \quad \eta^1 = \frac{c_2xs}{2} + \frac{c_1x}{2}, \quad \eta^2 = c_3t - c_6z + c_7,$$
$$\eta^3 = c_4t + c_6y + c_8, \quad \xi^0 = \frac{c_2s^2}{2} + c_1s + c_0, \quad A = \frac{-c_2x^2}{2} + c_9,$$
$$\nu(x) = \mu(x) = 2\ln\left(\frac{x}{\alpha}\right),$$

in system given in equations (4.1.13) we the get the following system of determining PDEs

$$
\begin{aligned}
&\xi^0_{a,t}=0,\quad \xi^0_{a,x}=0,\quad \xi^0_{a,y}=0,\quad \xi^0_{a,z}=0,\quad A_{a,s}=0,\\
&A_{a,t}-2\frac{x^2}{\alpha^2}\eta^0_{a,s}=0,\quad \frac{c_2tx}{\alpha}+A_{a,x}+2\eta^1_{a,s}=0,\\
&A_{a,y}+2\frac{x^2}{\alpha^2}\eta^2_{a,s}=0,\quad A_{a,z}+2\frac{x^2}{\alpha^2}\eta^3_{a,s}=0,\\
&\eta^1_{a,t}-\frac{x^2}{\alpha^2}\eta^0_{a,x}=0,\quad \eta^1_{a,y}+\frac{x^2}{\alpha^2}\eta^2_{a,x}=0,\\
&\eta^1_{a,z}+\frac{x^2}{\alpha^2}\eta^3_{a,x}=0,\quad \eta^2_{a,z}+\eta^3_{a,y}=0,\\
&\eta^0_{a,y}-\eta^2_{a,t}=0,\quad \eta^0_{a,z}-\eta^3_{a,t}=0,\\
&\big(c_3y+c_4z+c_5\big)f(t)+2\eta^1_{a,x}-\xi^0_{a,s}=0,\\
&\big(c_3y+c_4z+c_5\big)f_t(t)+\frac{2}{x}\eta^1_a+2\eta^0_{a,t}-\xi^0_{a,s}=0,\\
&\big(c_3y+c_4z+c_5\big)f_t(t)+\frac{2}{x}\eta^1_a+2\eta^2_{a,y}-\xi^0_{a,s}=0,\\
&\big(c_3y+c_4z+c_5\big)f_t(t)+\frac{2}{x}\eta^1_a+2\eta^3_{a,z}-\xi^0_{a,s}=0.
\end{aligned}
\tag{4.2.16}
$$

The solution of system given in equations (4.2.16) takes the form

$$
\begin{aligned}
&\eta^0_a=-\frac{c_2s\alpha^2}{4\alpha}+b_4y+b_5z+b_6,\quad \eta^1_a=\frac{b_2xs}{2}-\frac{c_3xy}{2\alpha}-\frac{c_4xz}{2\alpha}-\frac{c_5x}{2\alpha}+\frac{b_1x}{2}+,\\
&\eta^2_a=\frac{c_3\alpha^2\ln\frac{x}{\alpha}}{2\alpha}+b_4t-b_7z+b_8,\quad \eta^3_a=\frac{c_4\alpha^2\ln\frac{x}{\alpha}}{2\alpha}+b_5t+b_7y+b_9,\\
&A_a=-(b_2+\frac{c_2t}{\alpha})\frac{x^2}{2}+b_3,\quad \xi^0_a=\frac{b_2s^2}{2}+b_1s+b_0,\quad f(t)=\frac{t}{\alpha}.
\end{aligned}
\tag{4.2.17}
$$

The spacetime in this case takes the form

$$
\begin{aligned}
&ds^2=\left(\frac{x}{\alpha}\right)^2(dt^2-dy^2-dz^2)-dx^2+\\
&\frac{\epsilon t}{\alpha}\left(\left(\frac{x}{\alpha}\right)^2(dt^2-dy^2-dz^2)-dx^2\right),\quad \alpha\neq 0.
\end{aligned}
\tag{4.2.18}
$$

The approximate Noether symmetry generators are

$$
\begin{aligned}
&\mathbf{X_0}=\frac{\partial}{\partial t}-\frac{\epsilon x}{2\alpha}\frac{\partial}{\partial x},\quad \mathbf{X_1}=s^2\frac{\partial}{\partial s}+sx\frac{\partial}{\partial x}-\frac{\epsilon\alpha^2 s}{2\alpha}\frac{\partial}{\partial t},\quad A=-\frac{c_2t}{\alpha}x^2,\\
&\mathbf{X_2}=y\frac{\partial}{\partial t}+t\frac{\partial}{\partial y}-\frac{\epsilon}{2\alpha}\left(xy\frac{\partial}{\partial x}-\alpha^2\ln\frac{x}{\alpha}\frac{\partial}{\partial y}\right),\\
&\mathbf{X_3}=z\frac{\partial}{\partial t}+t\frac{\partial}{\partial z}-\frac{\epsilon}{2\alpha}\left(xz\frac{\partial}{\partial x}-\alpha^2\ln\frac{x}{\alpha}\frac{\partial}{\partial z}\right).
\end{aligned}
\tag{4.2.19}
$$

The first integrals corresponding to $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ are given in the following table

Table 4.4: First integrals

Gen	First integrals
$\mathbf{X_0}$	$\phi_0 = -2\left(\frac{x}{\alpha}\right)^2 \dot{t} - \frac{\epsilon}{\alpha}\left[\frac{2t\dot{t}}{\alpha}\left(\frac{x}{\alpha}\right)^2 + x\dot{x}\right]$
$\mathbf{X_1}$	$\phi_1 = s^2\mathcal{L} + 2sx\dot{x} + \frac{\epsilon}{\alpha}\left[s\alpha^2\dot{t}\left(\frac{x}{\alpha}\right)^2 + 2sxt\dot{x}\right]$
$\mathbf{X_2}$	$\phi_2 = 2\left(\frac{x}{\alpha}\right)^2(\dot{y}t - y\dot{t}) - \frac{\epsilon}{\alpha}\left[\left(\frac{x}{\alpha}\right)^2\left(2t\dot{t}y + \dot{y}\alpha^2 \ln\left(\frac{x}{\alpha}\right) - 2\dot{y}t^2\right) + x\dot{x}y\right]$
$\mathbf{X_3}$	$\phi_3 = 2\left(\frac{x}{\alpha}\right)^2(\dot{z}t - z\dot{t}) - \frac{\epsilon}{\alpha}\left[\left(\frac{x}{\alpha}\right)^2\left(2t\dot{t}z + \dot{z}b^2 \ln\left(\frac{x}{\alpha}\right) - 2\dot{z}t^2\right) + x\dot{x}z\right]$

Chapter 5

Noether Symmetries of the Arc Length Minimizing Lagrangian of Cylindrically Symmetric Static Spacetimes

5.1 Introduction

In this chapter symmetries of the arc length minimizing Lagrangian of cylindrically symmetric static spacetimes are given. These symmetries not only classify the spacetimes but also provide us first integrals corresponding to each Noether symmetry [1,9,20,33,43]. The first integrals give the conservation laws in the respective spacetimes. The classification of the spacetimes is carried out on the basis of different number of Noether symmetries, that action of the corresponding geodesic Lagrangian admit.

The most general form of cylindrically symmetric static spacetime is [14]

$$ds^2 = e^{\nu(r)}dt^2 - dr^2 - e^{\mu(r)}k^2 d\theta^2 - e^{\lambda(r)}dz^2. \tag{5.1.1}$$

The corresponding arc length minimizing Lagrangian density takes the form

$$\mathcal{L} = e^{\nu(r)}\dot{t}^2 - \dot{r}^2 - e^{\mu(r)}k^2\dot{\theta}^2 - e^{\lambda(r)}\dot{z}^2, \tag{5.1.2}$$

where " $\dot{}$ " denotes differentiation with respect to s. We obtain symmetries for the corresponding action of this Lagrangian density and then using the famous Noether's theorem we obtain first integral corresponding to each Noether symmetry.

5.2 Determining PDEs Of Cylindrically Symmetric Static Spacetimes

The Noether symmetry generator for the action of the arc length minimizing Lagrangian given by equation (5.1.2) is

$$\mathbf{X} = \xi \frac{\partial}{\partial s} + \eta^i \frac{\partial}{\partial x^i}, \tag{5.2.1}$$

and its first order prolongation is

$$\mathbf{X}^{[1]} = \mathbf{X} + \eta^i_s \frac{\partial}{\partial x^i}, \tag{5.2.2}$$

where x^i denote the dependent variables $t, r, \theta, z)$ and ξ, η^i [25] are functions of $(s, t, r, \theta, z.$ The components of the extended generator, that is η^i_s are functions of $s, t, r, \theta, z, \dot{t}, \dot{r}, \dot{\theta}, \dot{z}.$

Using the Lagrangian given by equation (5.1.2), symmetry generator given by equation (5.2.2) and differential operator given by equations (3.2.4) in the Noether symmetry equation (2.2.41) we get the following system of 19 PDEs

$$\begin{aligned}
&\xi_t = 0, \quad \xi_r = 0, \quad \xi_\theta = 0, \quad \xi_z = 0, \quad A_s = 0, \\
&2e^{\nu(r)}\eta^0_s = A_t, \quad -2\eta^1_s = A_r, \\
&-2k^2 e^{\mu(r)}\eta^2_s = A_\theta, \quad -2e^{\lambda(r)}\eta^3_s = A_z, \\
&\mu^{'}(r)\eta^1 + 2\eta^2_\theta - \xi_s = 0, \quad k^2 e^{\mu(r)}\eta^2_z - e^{\lambda(r)}\eta^3_\theta = 0, \\
&\eta^1_\theta + e^{\mu(r)}k^2\eta^2_r = 0, \quad \eta^1_z + e^{\lambda(r)}\eta^3_r = 0, \\
&e^{\nu(r)}\eta^0_r - \eta^1_t = 0, \quad e^{\nu(r)}\eta^0_\theta - e^{\mu(r)}k^2\eta^2_t = 0, \\
&e^{\nu(r)}\eta^0_z - e^{\lambda(r)}\eta^3_t = 0, \quad \nu^{'}(r)\eta^1 + 2\eta^0_t - \xi_s = 0, \\
&\lambda^{'}(r)\eta^1 + 2\eta^3_z - \xi_s = 0, \quad 2\eta^1_r - \xi_s = 0.
\end{aligned} \tag{5.2.3}$$

Solutions of this system give us distinct arc length minimizing Lagrangian densities of cylindrically symmetric static spacetimes along with distinct set of the Noether symmetries and first integrals. In order to keep a distinction between Killing vector fields and Noether symmetries we use a different letter, namely, $\mathbf{Y}$ for those Noether symmetries which are not Killing vector fields. It is also remarked that the Lagrangian given by equation (5.1.2) is independent of s, t, θ and z, so the minimal set of the Noether symmetries admitting by the action of the geodesic Lagrangian density given by equation (5.1.2) of cylindrically symmetric static spacetime consists of four Noether symmetries which are

$$\mathbf{X_0} = \frac{\partial}{\partial t}, \qquad \mathbf{X_1} = \frac{\partial}{\partial \theta}, \qquad \mathbf{X_2} = \frac{\partial}{\partial z}, \quad \mathbf{Y_0} = \frac{\partial}{\partial s}. \tag{5.2.4}$$

Here $\mathbf{X_0}$, $\mathbf{X_1}$, $\mathbf{X_2}$ are isometries and $\mathbf{Y_0}$ is the Noether symmetry under which the Lagrangian density remains invariant. For each Noether symmetry a first integral can be obtained by equation (3.2.5), where x^i denotes the dependent variables t, r, θ, z. First integrals corresponding to the minimal set of Noether symmetries for arc length minimizing Lagrangian of cylindrically symmetric static spacetime are given in Table 5.1.

Table 5.1: First Integrals for the minimal set

Gen	First Integrals
$\mathbf{X_0}$	$\phi_0 = -2e^{\nu(r)}\dot{t}$
$\mathbf{X_1}$	$\phi_1 = 2e^{\mu(r)}k^2\dot{\theta}$
$\mathbf{X_2}$	$\phi_2 = 2e^{\lambda(r)}\dot{z}$
$\mathbf{Y_0}$	$\phi_3 = e^{\nu(r)}\dot{t}^2 - \dot{r}^2 - e^{\mu(r)}k^2\dot{\theta}^2 - e^{\lambda(r)}\dot{z}^2 = \mathcal{L}$

5.3 Five Noether Symmetries and First Integrals

There are infinitely many classes of cylindrically symmetric static spacetimes for which the action of the corresponding Lagrangians admit five Noether symmetries. We list some of the classes of these spacetimes in Table 5.2. List of the fifth symmetry and the corresponding first integrals for the classes given in Table 5.2 are given in Table 5.3.

Table 5.2: Cases of Five symmetries

No.	$\nu(r)$	$\mu(r)$	$\lambda(r)$
1.	$a\ln\left(\frac{r}{\alpha}\right)$	$\frac{r}{\alpha}$	const
2.	$a\ln\left(\frac{r}{\alpha}\right)$	$\mu(r)\neq b\ln\left(\frac{r}{\alpha}\right)$	const
3.	$\frac{r}{\alpha}$	$2\ln\left(\frac{r}{\alpha}\right)$	const
4.	$\nu(r)\neq a\ln\left(\frac{r}{\alpha}\right)$	$b\ln\left(\frac{r}{\alpha}\right)$	const
5.	$a\ln\left(\frac{r}{\alpha}\right)$	const	$\frac{r}{\alpha}$
6.	$a\ln\left(\frac{r}{\alpha}\right)$	const	$\lambda(r)\neq c\ln\left(\frac{r}{\alpha}\right)$
7.	$\frac{r}{\alpha}$	const	$b\ln\left(\frac{r}{\alpha}\right)$
8.	$\nu(r)\neq a\ln\left(\frac{r}{\alpha}\right)$	const	$c\ln\left(\frac{r}{\alpha}\right)$
9.	const	$b\ln\left(\frac{r}{\alpha}\right)$	$\lambda(r)\neq c\ln\left(\frac{r}{\alpha}\right)$
10.	const	$b\ln\left(\frac{r}{\alpha}\right)$	$\frac{r}{\alpha}$
11.	const	$\frac{r}{\alpha}$	$c\ln\left(\frac{r}{\alpha}\right)$
12.	$\frac{r}{\alpha}$	$\frac{r}{\beta}$	$\frac{r}{\gamma}$
13.	$2\ln\left(\frac{r}{\alpha}\right)$	$b\ln\left(\frac{r}{\beta}\right)$	$c\ln\left(\frac{r}{\gamma}\right)$
14.	$a\ln\left(\frac{r}{\alpha}\right)$	$2\ln\left(\frac{r}{\beta}\right)$	$c\ln\left(\frac{r}{\gamma}\right)$
15.	$a\ln\left(\frac{r}{\alpha}\right)$	$b\ln\left(\frac{r}{\beta}\right)$	$2\ln\left(\frac{r}{\gamma}\right)$
16.	$a\ln\left(\frac{r}{\alpha}\right)$	$b\ln\left(\frac{r}{\beta}\right)$	$c\ln\left(\frac{r}{\gamma}\right)$

5.4 Six Noether Symmetries and First Integrals

In Table 5.4, a list of coefficients of cylindrically symmetric static spacetime, additional Noether symmetries and gauge functions are given. There are 24 classes of cylindrically symmetric static spacetimes the actions of the corresponding geodesic Lagrangians of which admit six Noether symmetries. The first integrals or conservation laws corresponding to the symmetries given in Table 5.4 are given in Table 5.5. It is evident form Table 5.3 that the fifth Noether symmetry generator for arc length minimizing Lagrangian densities of cylindrically symmetric static spacetimes is either a galilean symmetries or homothety. We also see from Table 5.4 that the fifth and sixth Noether symmetries are homothety and galilean or homothety and rotations (pure rotation or boosts).

Table 5.3: Cases of Five Symmetries

No.	Fifth symmetry	Gauge Functions	First Integral
1.	$\mathbf{Y_1} = s\frac{\partial}{\partial z}$	$A = -2z$	$\phi_4 = 2[s\dot{z} - z]$
2.	$\mathbf{Y_1} = s\frac{\partial}{\partial z}$	$A = -2z$	$\phi_4 = 2[s\dot{z} - z]$
3.	$\mathbf{Y_1} = s\frac{\partial}{\partial z}$	$A = -2z$	$\phi_4 = 2[s\dot{z} - z]$
4.	$\mathbf{Y_1} = s\frac{\partial}{\partial z}$	$A = -2z$	$\phi_4 = 2[s\dot{z} - z]$
5.	$\mathbf{Y_1} = s\frac{\partial}{\partial \theta}$	$A = -2\theta$	$\phi_4 = 2k^2[s\dot{\theta} - \theta]$
6.	$\mathbf{Y_1} = s\frac{\partial}{\partial \theta}$	$A = -2\theta$	$\phi_4 = 2k^2[s\dot{\theta} - \theta]$
7.	$\mathbf{Y_1} = s\frac{\partial}{\partial \theta}$	$A = -2\theta$	$\phi_4 = 2k^2[s\dot{\theta} - \theta]$
8.	$\mathbf{Y_1} = s\frac{\partial}{\partial \theta}$	$A = -2\theta$	$\phi_4 = 2k^2[s\dot{\theta} - \theta]$
9.	$\mathbf{Y_1} = s\frac{\partial}{\partial t}$	$A = 2t$	$\phi_4 = 2[t - s\dot{t}]$
10.	$\mathbf{Y_1} = s\frac{\partial}{\partial t}$	$A = 2t$	$\phi_4 = 2[t - s\dot{t}]$
11.	$\mathbf{Y_1} = s\frac{\partial}{\partial t}$	$A = 2t$	$\phi_4 = 2[t - s\dot{t}]$
12.	$\mathbf{X_3} = \frac{\partial}{\partial r} - \frac{t}{2\alpha}\frac{\partial}{\partial t} - \frac{\theta}{2\beta}\frac{\partial}{\partial \theta} - \frac{z}{2\gamma}\frac{\partial}{\partial z}$	$A = 0$	$\phi_4 = 2\dot{r} + \frac{t\dot{t}e^{\frac{r}{\alpha}}}{\alpha} - \frac{\theta\dot{\theta}k^2e^{\frac{r}{\beta}}}{\beta} - \frac{z\dot{z}e^{\frac{r}{\gamma}}}{\gamma}$
13.	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-b}{4}\theta\frac{\partial}{\partial \theta} + \frac{2-c}{4}z\frac{\partial}{\partial z}$	$A = 0$	$\phi_4 = s\mathcal{L} + \frac{(2-b)\theta\dot{\theta}k^2}{2}\left(\frac{r}{\alpha}\right)^b + \frac{(2-c)z\dot{z}}{2}\left(\frac{r}{\alpha}\right)^c$
14.	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-a}{4}t\frac{\partial}{\partial t} + \frac{2-c}{4}z\frac{\partial}{\partial z}$	$A = 0$	$\phi_4 = s\mathcal{L} - \frac{(2-a)t\dot{t}}{2}\left(\frac{r}{\alpha}\right)^a + \frac{(2-c)z\dot{z}}{2}\left(\frac{r}{\alpha}\right)^c$
15.	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-a}{4}t\frac{\partial}{\partial t} + \frac{2-b}{4}\theta\frac{\partial}{\partial \theta}$	$A = 0$	$\phi_4 = s\mathcal{L} - \frac{(2-a)t\dot{t}}{2}\left(\frac{r}{\alpha}\right)^a + \frac{(2-b)\theta\dot{\theta}k^2}{2}\left(\frac{r}{\alpha}\right)^b$
16.	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-a}{4}t\frac{\partial}{\partial t} + \frac{2-b}{4}\theta\frac{\partial}{\partial \theta} + \frac{2-c}{4}z\frac{\partial}{\partial z}$	$A = 0$	$\phi_4 = s\mathcal{L} - \frac{(2-a)t\dot{t}}{2}\left(\frac{r}{\alpha}\right)^a + \frac{(2-b)\theta\dot{\theta}k^2}{2}\left(\frac{r}{\alpha}\right)^b + \frac{(2-c)z\dot{z}}{2}\left(\frac{r}{\alpha}\right)^c$

5.5 Seven Noether Symmetries and First Integrals

All classes of cylindrically symmetric static spacetimes for which the action of arc length minimizing Lagrangians admit seven Noether symmetries are given in this section. There are three classes of cylindrically symmetric static spacetime the actions of the Lagrangians of which admit seven Noether symmetries, the detail of these spacetimes are as foolows :

Solution-I:

Coefficient of the metric are

$$\nu(r) = c, \quad \mu(r) = b\ln\left(\frac{r}{\alpha}\right) = \lambda(r).$$

Table 5.4: Cases of Six Symmetries

No.	Metrics	Symmetries, Gauge Functions
1.	$\nu(r)=\frac{r}{\alpha}, \mu(r)=const, \lambda(r)=\frac{r}{\beta}$	$\mathbf{X_3}=\frac{\partial}{\partial r}-\frac{t}{2\alpha}\frac{\partial}{\partial t}-\frac{z}{2\beta}\frac{\partial}{\partial z}, \mathbf{Y_1}=s\frac{\partial}{\partial\theta}, A_1=-2k^2\theta$
2.	$\nu(r)=\frac{r}{\alpha}, \mu(r)=\frac{r}{\beta}, \lambda(r)=const$	$\mathbf{X_3}=\frac{\partial}{\partial r}-\frac{t}{2\alpha}\frac{\partial}{\partial t}-\frac{\theta}{2\beta}\frac{\partial}{\partial\theta}, \mathbf{Y_1}=s\frac{\partial}{\partial z}, A_1=-2z$
3.	$\nu(r)=const, \mu(r)=\frac{r}{\alpha}, \lambda(r)=\frac{r}{\beta}$	$\mathbf{X_3}=\frac{\partial}{\partial r}-\frac{\theta}{2\alpha}\frac{\partial}{\partial\theta}-\frac{z}{2\alpha}\frac{\partial}{\partial z}, \mathbf{Y_1}=s\frac{\partial}{\partial t}, A_1=2t$
4.	$\nu(r)=\frac{r}{\alpha}, \mu(r)=\frac{r}{\beta}, \lambda(r)=\frac{r}{\alpha}$	$\mathbf{X_3}=\frac{\partial}{\partial r}-\frac{t}{2\alpha}\frac{\partial}{\partial t}-\frac{\theta}{2\beta}\frac{\partial}{\partial\theta}-\frac{z}{2\alpha}\frac{\partial}{\partial z}, \mathbf{X_4}=z\frac{\partial}{\partial t}+t\frac{\partial}{\partial z}$
5.	$\nu(r)=\frac{r}{\alpha}, \mu(r)=\frac{r}{\beta}, \lambda(r)=\frac{r}{\beta}$	$\mathbf{X_3}=\frac{\partial}{\partial r}-\frac{t}{2\alpha}\frac{\partial}{\partial t}-\frac{\theta}{2\beta}\frac{\partial}{\partial\theta}-\frac{z}{2\beta}\frac{\partial}{\partial z}, \mathbf{X_4}=z\frac{\partial}{\partial\theta}-\theta k^2\frac{\partial}{\partial z}$
6.	$\nu(r)=\frac{r}{\alpha}, \mu(r)=\frac{r}{\alpha}, \lambda(r)=\frac{r}{\beta}$	$\mathbf{X_3}=\frac{\partial}{\partial r}-\frac{t}{2\alpha}\frac{\partial}{\partial t}-\frac{\theta}{2\alpha}\frac{\partial}{\partial\theta}-\frac{z}{2\beta}\frac{\partial}{\partial z}, \mathbf{X_4}=t\frac{\partial}{\partial\theta}+\theta k^2\frac{\partial}{\partial t}$
7.	$\nu(r)=a\ln\frac{r}{\alpha}, \mu(r)=2\ln\frac{r}{\alpha}, \lambda(r)=const$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{2-a}{4}t\frac{\partial}{\partial t}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{z}{2}\frac{\partial}{\partial z}, \quad \mathbf{Y_2}=s\frac{\partial}{\partial z}, A_2=-2z$
8.	$\nu(r)=2\ln\frac{r}{\alpha}, \mu(r)=b\ln\frac{r}{\alpha}, \lambda(r)=const$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{2-b}{4}\theta\frac{\partial}{\partial\theta}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{z}{2}\frac{\partial}{\partial z}, \mathbf{Y_2}=s\frac{\partial}{\partial z}, \quad A_2=-2z$
9.	$\nu(r)=a\ln\frac{r}{\alpha}, \mu(r)=b\ln\frac{r}{\alpha}, \lambda(r)=const$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{2-a}{4}t\frac{\partial}{\partial t}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{2-b}{4}\theta\frac{\partial}{\partial\theta}+\frac{z}{2}\frac{\partial}{\partial z}, \mathbf{Y_2}=s\frac{\partial}{\partial z}, A_2=-2z$
10.	$\nu(r)=const, \mu(r)=2\ln\frac{r}{\alpha}, \lambda(r)=c\ln\frac{r}{\alpha}$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{t}{2}\frac{\partial}{\partial t}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{2-c}{4}z\frac{\partial}{\partial z}, \mathbf{Y_2}=s\frac{\partial}{\partial t}, A_2=2t$
11.	$\nu(r)=const, \mu(r)=b\ln\frac{r}{\alpha}, \lambda(r)=2\ln\frac{r}{\alpha}$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{t}{t}\frac{\partial}{\partial t}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{2-b}{4}\theta\frac{\partial}{\partial\theta}, \mathbf{Y_2}=s\frac{\partial}{\partial t}, A_2=2t$
12.	$\nu(r)=const, \mu(r)=b\ln\frac{r}{\alpha}, \lambda(r)=c\ln\frac{r}{\alpha}$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{t}{2}\frac{\partial}{\partial t}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{2-b}{4}\theta\frac{\partial}{\partial\theta}+\frac{2-c}{4}z\frac{\partial}{\partial z}, \mathbf{Y_2}=s\frac{\partial}{\partial z}, A_2=2t$
13.	$\nu(r)=2\ln\frac{r}{\alpha}, \mu(r)=const, \lambda(r)=c\ln\frac{r}{\alpha}$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{\theta}{2}\frac{\partial}{\partial\theta}+\frac{2-c}{4}z\frac{\partial}{\partial z}, \mathbf{Y_2}=s\frac{\partial}{\partial\theta}, A_2=-2k^2\theta$
14.	$\nu(r)=a\ln\frac{r}{\alpha}, \mu(r)=const, \lambda(r)=2\ln\frac{r}{\alpha}$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{2-a}{4}t\frac{\partial}{\partial t}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{\theta}{2}\frac{\partial}{\partial\theta}, \mathbf{Y_2}=s\frac{\partial}{\partial\theta}, A_2=-2k^2\theta$
15.	$\nu(r)=a\ln\frac{r}{\alpha}, \mu(r)=const, \lambda(r)=c\ln\frac{r}{\alpha}$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{2-a}{4}t\frac{\partial}{\partial t}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{\theta}{2}\frac{\partial}{\partial\theta}+\frac{2-c}{4}z\frac{\partial}{\partial z}, \mathbf{Y_2}=s\frac{\partial}{\partial\theta}, A_2=-2k^2\theta$
16.	$\nu(r)=a\ln\frac{r}{\alpha}, \mu(r)=2\ln\frac{r}{\alpha}, \lambda(r)=a\ln\frac{r}{\alpha}$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{2-a}{4}t\frac{\partial}{\partial t}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{2-a}{4}z\frac{\partial}{\partial z}, \quad \mathbf{X_3}=z\frac{\partial}{\partial t}+t\frac{\partial}{\partial z}$
17.	$\nu(r)=2\ln\frac{r}{\alpha}, \mu(r)=b\ln\frac{r}{\alpha}, \lambda(r)=b\ln\frac{r}{\alpha}$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{2-b}{4}\theta\frac{\partial}{\partial\theta}+\frac{2-b}{4}z\frac{\partial}{\partial z}, \mathbf{X_3}=z\frac{\partial}{\partial\theta}-\theta k^2\frac{\partial}{\partial z}$
18.	$\nu(r)=a\ln\frac{r}{\alpha}, \mu(r)=a\ln\frac{r}{\alpha}, \lambda(r)=2\ln\frac{r}{\alpha}$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{2-a}{4}t\frac{\partial}{\partial t}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{2-a}{4}\theta\frac{\partial}{\partial\theta}, \mathbf{X_3}=k^2\theta\frac{\partial}{\partial\theta}+t\frac{\partial}{\partial\theta}$
19.	$\nu(r)=2\ln\frac{r}{\alpha}, \mu(r)=b\ln\frac{r}{\alpha}, \lambda(r)=2\ln\frac{r}{\alpha}$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{2-b}{4}\theta\frac{\partial}{\partial\theta}, \mathbf{X_3}=z\frac{\partial}{\partial t}+t\frac{\partial}{\partial z}$
20.	$\nu(r)=2\ln\frac{r}{\alpha}, \mu(r)=2\ln\frac{r}{\alpha}, \lambda(r)=c\ln\frac{r}{\alpha}$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{2-c}{4}z\frac{\partial}{\partial z}, \mathbf{X_3}=k^2\theta\frac{\partial}{\partial t}+t\frac{\partial}{\partial\theta}$
21.	$\nu(r)=a\ln\frac{r}{\alpha}, \mu(r)=2\ln\frac{r}{\alpha}, \lambda(r)=2\ln\frac{r}{\alpha}$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{2-a}{4}t\frac{\partial}{\partial t}, \mathbf{X_3}=z\frac{\partial}{\partial\theta}-k^2\theta\frac{\partial}{\partial z}$
22.	$\nu(r)=a\ln\frac{r}{\alpha}, \mu(r)=a\ln\frac{r}{\alpha}, \lambda(r)=c\ln\frac{r}{\alpha}$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{2-a}{4}t\frac{\partial}{\partial t}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{2-a}{4}\theta\frac{\partial}{\partial\theta}+\frac{2-c}{4}z\frac{\partial}{\partial z}, \mathbf{X_3}=k^2\theta\frac{\partial}{\partial t}+t\frac{\partial}{\partial\theta}$
23.	$\nu(r)=a\ln\frac{r}{\alpha}, \mu(r)=b\ln\frac{r}{\alpha}, \lambda(r)=a\ln\frac{r}{\alpha}$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{2-a}{4}t\frac{\partial}{\partial t}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{2-b}{4}\theta\frac{\partial}{\partial\theta}+\frac{2-a}{4}z\frac{\partial}{\partial z}, \mathbf{X_3}=z\frac{\partial}{\partial t}+t\frac{\partial}{\partial z}$
24.	$\nu(r)=a\ln\frac{r}{\alpha}, \mu(r)=b\ln\frac{r}{\alpha}, \lambda(r)=b\ln\frac{r}{\alpha}$	$\mathbf{Y_1}=s\frac{\partial}{\partial s}+\frac{2-a}{4}t\frac{\partial}{\partial t}+\frac{r}{2}\frac{\partial}{\partial r}+\frac{2-b}{4}\theta\frac{\partial}{\partial\theta}+\frac{2-b}{4}z\frac{\partial}{\partial z}, \mathbf{X_3}=z\frac{\partial}{\partial\theta}-k^2\theta\frac{\partial}{\partial z}$

Table 5.5: First Integrals For Table 5.4

No.	Generators	First Integrals
1.	$\mathbf{X_3}$, $\mathbf{Y_1}$	$\phi_4 = 2\dot{r} + \frac{e^{\frac{r}{\alpha}} t\dot{t}}{\alpha} - \frac{e^{\frac{r}{\beta}} z\dot{z}}{\beta}$, $\phi_5 = 2k^2[s\dot{\theta} - \theta]$
2.	$\mathbf{X_3}$, $\mathbf{Y_1}$	$\phi_4 = 2\dot{r} + \frac{e^{\frac{r}{\alpha}} t\dot{t}}{\alpha} - \frac{e^{\frac{r}{\beta}} \theta\dot{\theta}k^2}{\beta}$, $\phi_5 = 2[s\dot{z} - z]$
3.	$\mathbf{X_3}$, $\mathbf{Y_1}$	$\phi_4 = 2\dot{r} - \frac{e^{\frac{r}{\alpha}} \theta\dot{\theta}k^2}{\alpha} - \frac{e^{\frac{r}{\beta}} z\dot{z}}{\beta}$, $\phi_5 = 2[t - s\dot{t}]$
4.	$\mathbf{X_3}$, $\mathbf{X_4}$	$\phi_4 = 2\dot{r} + \frac{e^{\frac{r}{\beta}} t\dot{t}}{\beta} - \frac{e^{\frac{r}{\beta}} \theta\dot{\theta}k^2}{\beta} - \frac{e^{\frac{r}{\beta}} z\dot{z}}{\beta}$, $\phi_5 = 2e^{\frac{r}{\alpha}}[t\dot{z} - z\dot{t}]$
5.	$\mathbf{X_3}$, $\mathbf{X_4}$	$\phi_4 = 2\dot{r} + \frac{e^{\frac{r}{\beta}} t\dot{t}}{\beta} - \frac{e^{\frac{r}{\beta}} \theta\dot{\theta}k^2}{\beta} - \frac{e^{\frac{r}{\beta}} z\dot{z}}{\beta}$, $\phi_5 = 2e^{\frac{r}{\beta}}k^2[z\dot{\theta} - \theta\dot{z}]$
6.	$\mathbf{X_3}$, $\mathbf{X_4}$	$\phi_4 = 2\dot{r} + \frac{e^{\frac{r}{\beta}} t\dot{t}}{\beta} - \frac{e^{\frac{r}{\alpha}} \theta\dot{\theta}k^2}{\alpha} - \frac{e^{\frac{r}{\beta}} z\dot{z}}{\beta}$, $\phi_5 = 2e^{\frac{r}{\alpha}}k^2[t\dot{\theta} - \theta\dot{t}]$
7.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = s\mathcal{L} - \frac{2-a}{2}\left(\frac{r}{\alpha}\right)^a t\dot{t} + r\dot{r} + z\dot{z}$, $\phi_5 = 2[s\dot{z} - z]$
8.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = s\mathcal{L} + \frac{2-a}{2}\left(\frac{r}{\alpha}\right)^a \theta\dot{\theta}k^2 + r\dot{r} + z\dot{z}$, $\phi_5 = 2[s\dot{z} - z]$
9.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = s\mathcal{L} - \frac{2-a}{2}\left(\frac{r}{\alpha}\right)^a t\dot{t} + \frac{2-l}{2}\left(\frac{r}{\alpha}\right)^b \theta\dot{\theta}k^2 + r\dot{r} + z\dot{z}$, $\phi_5 = 2[s\dot{z} - z]$
10.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = s\mathcal{L} - t\dot{t} + r\dot{r} + \frac{2-c}{2}\left(\frac{r}{\alpha}\right)^c z\dot{z}$, $\phi_5 = 2[t - s\dot{t}]$
11.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = s\mathcal{L} - t\dot{t} + r\dot{r} + \frac{2-b}{2}\left(\frac{r}{\alpha}\right)^b \theta\dot{\theta}k^2$, $\phi_5 = 2[t - s\dot{t}]$
12.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = s\mathcal{L} - t\dot{t} + \frac{2-b}{2}\left(\frac{r}{\alpha}\right)^2 \theta\dot{\theta}k^2 + r\dot{r} + \frac{2-c}{2}\left(\frac{r}{\alpha}\right)^c z\dot{z}$, $\phi_5 = 2[-s\dot{t} + t]$
13.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = s\mathcal{L} + r\dot{r} + \theta\dot{\theta}k^2 + \frac{2-c}{2}\left(\frac{r}{\alpha}\right)^c z\dot{z}$, $\phi_5 = 2k^2[s\dot{\theta} - \theta]$
14.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = s\mathcal{L} - \frac{2-a}{2}\left(\frac{r}{\alpha}\right)^a t\dot{t} + r\dot{r} + \theta\dot{\theta}k^2$, $\phi_5 = 2k^2[s\dot{\theta} - \theta]$
15.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = s\mathcal{L} - \frac{2-a}{2}\left(\frac{r}{\alpha}\right)^a t\dot{t} + r\dot{r} + \theta\dot{\theta}k^2 + \frac{2-c}{2}\left(\frac{r}{\alpha}\right)^c z\dot{z}$, $\phi_5 = 2k^2[s\dot{\theta} - \theta]$
16.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = s\mathcal{L} - \frac{2-a}{2}\left(\frac{r}{\alpha}\right)^a t\dot{t} + r\dot{r}$, $\phi_5 = 2[t\dot{z} - z\dot{t}]$
17.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = s\mathcal{L} + r\dot{r} + \left(\frac{r}{\alpha}\right)^2 t\dot{t} + \frac{2-b}{2}\left(\frac{r}{\alpha}\right)^b \theta\dot{\theta}k^2 + \frac{2-b}{2}\left(\frac{r}{\alpha}\right)^b z\dot{z}$, $\phi_5 = 2k^2[z\dot{\theta} - \theta\dot{z}]$
18.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = s\mathcal{L} - \frac{2-a}{2}\left(\frac{r}{\alpha}\right)^a t\dot{t} + r\dot{r} + \frac{2-a}{2}\left(\frac{r}{\alpha}\right)^a \theta\dot{\theta}k^2$, $\phi_5 = 2k^2[t\dot{\theta} - \theta\dot{t}]$
19.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = s\mathcal{L} + r\dot{r} + \frac{2-b}{2}\left(\frac{r}{\alpha}\right)^b \theta\dot{\theta}k^2$, $\phi_5 = 2[t\dot{z} - z\dot{t}]$
20.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = s\mathcal{L} + r\dot{r} + \frac{2-c}{2}\left(\frac{r}{\alpha}\right)^c z\dot{z}$, $\phi_5 = 2k^2[t\dot{\theta} - \theta\dot{t}]$
21.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = s\mathcal{L} + r\dot{r} - \frac{2-a}{2}\left(\frac{r}{\alpha}\right)^a t\dot{t}$, $\phi_5 = 2k^2[z\dot{\theta} - \theta\dot{z}]$
22.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = s\mathcal{L} + r\dot{r} + \frac{2-a}{2}\left(\frac{r}{\alpha}\right)^a t\dot{t} + \frac{2-a}{2}\left(\frac{r}{\alpha}\right)^a \theta\dot{\theta}k^2 + \frac{2-c}{2}\left(\frac{r}{\alpha}\right)^c z\dot{z}$, $\phi_5 = 2k^2[t\dot{\theta} - \theta\dot{t}]$
23.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = s\mathcal{L} + r\dot{r} + \frac{2-a}{2}\left(\frac{r}{\alpha}\right)^a t\dot{t} + \frac{2-b}{2}\left(\frac{r}{\alpha}\right)^b \theta\dot{\theta}k^2 + \frac{2-a}{2}\left(\frac{r}{\alpha}\right)^a z\dot{z}$, $\phi_5 = 2[t\dot{z} - z\dot{t}]$
24.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = s\mathcal{L} + r\dot{r} - \frac{2-a}{2}\left(\frac{r}{\alpha}\right)^a t\dot{t} + \frac{2-b}{2}\left(\frac{r}{\alpha}\right)^b \theta\dot{\theta}k^2 + \frac{2-c}{2}\left(\frac{r}{\alpha}\right)^c z\dot{z}$, $\phi_5 = 2k^2[z\dot{\theta} - \theta\dot{z}]$

Components of the Noether symmetry generators are

$$\xi = c_1 s + c_2, \quad \eta^0 = c_1\frac{t}{2} + c_3 s + c_4, \quad \eta^1 = c_1\frac{r}{2}, \quad \eta^2 = c_1\frac{2-b}{4}\theta - c_5 z + c_6,$$
$$\eta^3 = c_1\frac{2-b}{4}z + c_5 k^2\theta + c_7, \quad A = 2c_3 t + c_8.$$

The spacetime for Solution-I above takes the form

$$ds^2 = dt^2 - dr^2 - \left(\frac{r}{\alpha}\right)^b k^2 d\theta^2 - \left(\frac{r}{\alpha}\right)^b dz^2, \quad k \neq 0, 2, \quad \alpha \neq 0.$$

The additional symmetries along with set of symmetries in equation (5.2.4) are

$$\begin{aligned} \mathbf{Y_1} &= s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-b}{4}\theta\frac{\partial}{\partial \theta} + \frac{2-b}{4}z\frac{\partial}{\partial z}, \\ \mathbf{Y_2} &= s\frac{\partial}{\partial t}, \quad A_2 = 2t, \qquad \mathbf{X_3} = -z\frac{\partial}{\partial \theta} + k^2\theta\frac{\partial}{\partial z}. \end{aligned} \tag{5.5.1}$$

The first integrals of these symmetries are given in Table 5.6.

Table 5.6: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\left(\frac{r}{\alpha}\right)^b k^2[\theta\dot{z} - z\dot{\theta}]$
$\mathbf{Y_1}$	$\phi_5 = s\mathcal{L} - t\dot{t} + r\dot{r} + \left(\frac{r}{\alpha}\right)^b \frac{2-b}{2}\theta\dot{\theta}k^2 + \left(\frac{r}{\alpha}\right)^b \frac{2-b}{2} z\dot{z}$
$\mathbf{Y_2}$	$\phi_6 = 2[t - s\dot{t}]$

Solution-II:

Coefficients of the metric are

$$\nu(r) = \mu(r) = a\ln\left(\frac{r}{\alpha}\right), \quad \lambda(r) = c.$$

Components of the Noethter symmetry generators are

$$\begin{aligned} &\xi = c_1 s + c_2, \quad \eta^0 = c_1\frac{2-a}{4}t + c_3k^2\theta + c_4, \quad \eta^1 = c_1\frac{r}{2}, \quad \eta^2 = c_1\frac{2-a}{4}\theta + c_3 t + c_6, \\ &\eta^2 = c_5 s + c_1\frac{z}{2} + c_7, \quad A = -c_5 2z + c_8. \end{aligned}$$

The spacetime takes the form

$$ds^2 = \left(\frac{r}{\alpha}\right)^a dt^2 - dr^2 - \left(\frac{r}{\alpha}\right)^a k^2 d\theta^2 - dz^2, \quad k \neq 0, 2, \quad \alpha \neq 0.$$

Here we have the following three additional symmetries along with the minimal set.

$$\begin{aligned} \mathbf{Y_1} &= s\frac{\partial}{\partial s} + \frac{2-a}{4}t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-a}{4}\theta\frac{\partial}{\partial \theta} + \frac{z}{2}\frac{\partial}{\partial z}, \\ \mathbf{Y_2} &= s\frac{\partial}{\partial z}, \qquad A_2 = -2z, \qquad \mathbf{X_3} = k^2\theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta}. \end{aligned} \tag{5.5.2}$$

The first integrals are given in the following Table 5.7.

Table 5.7: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\left(\frac{r}{\alpha}\right)^a k^2[t\dot{\theta} - \theta\dot{t}]$
$\mathbf{Y_1}$	$\phi_5 = s\mathcal{L} - \left(\frac{r}{\alpha}\right)^a \frac{2-k}{2} t\dot{t} + r\dot{r} + \left(\frac{r}{\alpha}\right)^a \frac{2-a}{2} \theta\dot{\theta}k^2 + z\dot{z}$
$\mathbf{Y_2}$	$\phi_6 = 2[s\dot{z} - z]$

Solution-III:

The metric coefficients are

$$\nu(r) = \lambda(r) = a \ln\left(\frac{r}{\alpha}\right), \quad \mu(r) = c.$$

Components of the Noethter symmetry generators are

$$\xi = c_1 s + c_2, \quad \eta^0 = c_1 \frac{2-a}{4} t + c_3 z + c_4, \quad \eta^1 = c_1 \frac{r}{2}, \quad \eta^2 = c_1 \frac{\theta}{2} + c_5 s + c_6,$$
$$\eta^2 = c_3 t + c_1 \frac{2-a}{4} z + c_7, \quad A = -c_5 2k^2\theta + c_8.$$

The spacetime takes the form

$$ds^2 = \left(\frac{r}{\alpha}\right)^a dt^2 - dr^2 - k^2 d\theta^2 - \left(\frac{r}{\alpha}\right)^a dz^2, \quad a \neq 0, 2, \quad \alpha \neq 0. \tag{5.5.3}$$

The following are the additional Noether symmetries

$$\begin{aligned} \mathbf{Y_1} &= s\frac{\partial}{\partial s} + \frac{2-a}{4} t \frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{\theta}{2}\frac{\partial}{\partial \theta} + \frac{2-a}{4} z \frac{\partial}{\partial z}, \\ \mathbf{Y_2} &= s\frac{\partial}{\partial \theta}, \qquad A_2 = -2k^2\theta, \qquad \mathbf{X_3} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}. \end{aligned} \tag{5.5.4}$$

The first integrals for these symmetries are given in Table 5.8.

Table 5.8: First integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\left(\frac{r}{\alpha}\right)^a [t\dot{z} - z\dot{t}]$
$\mathbf{Y_1}$	$\phi_5 = s\mathcal{L} - \left(\frac{r}{\alpha}\right)^a \frac{2-a}{2} t\dot{t} + r\dot{r} + \theta\dot{\theta}k^2 + \left(\frac{r}{\alpha}\right)^a \frac{2-a}{2} z\dot{z}$
$\mathbf{Y_2}$	$\phi_6 = 2k^2[s\dot{\theta} - \theta]$

5.6 Eight Noether Symmetries and First Integrals

There are seven different classes of cylindrically symmetric static spacetimes the actions of the Lagrangians of which admit eight Noether symmetries. The detail of these spacetimes is given in this section:

Solution-I:

The metric of the spacetime has the following coefficients

$$\nu(r) = a\ln\left(\frac{r}{\alpha}\right), \quad \mu(r) = c, \quad \lambda(r) = c_0.$$

Components of the Noether symmetry generators are

$$\xi = c_1 s + c_2, \quad \eta^0 = c_1\frac{2-a}{4}t + c_3, \quad \eta^1 = c_1\frac{r}{2}, \quad \eta^2 = c_1\frac{\theta}{2} + c_4 s - c_5 z + c_6,$$
$$\eta^3 = c_1\frac{z}{2} + c_7 s + c_5 k^2\theta + c_8, \quad A = -c_4 2k^2\theta - c_7 2z + c_9.$$

The spacetime takes the form

$$ds^2 = \left(\frac{r}{\alpha}\right)^a dt^2 - dr^2 - k^2 d\theta^2 - dz^2, \quad a \neq 0, 2, \quad \alpha \neq 0. \tag{5.6.1}$$

The additional symmetries are

$$\begin{aligned} &\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-a}{4}t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{\theta}{2}\frac{\partial}{\partial \theta} + \frac{z}{2}\frac{\partial}{\partial z}, \quad \mathbf{Y_2} = s\frac{\partial}{\partial \theta}, \quad A_2 = -2k^2\theta, \\ &\mathbf{Y_3} = s\frac{\partial}{\partial z}, \qquad A_3 = -2z, \qquad \mathbf{X_3} = z\frac{\partial}{\partial \theta} - k^2\theta\frac{\partial}{\partial z}. \end{aligned} \tag{5.6.2}$$

The corresponding first integral are given in Table 5.9.

Table 5.9: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2k^2[z\dot{\theta} - \theta\dot{z}]$
$\mathbf{Y_1}$	$\phi_5 = s\mathcal{L} - \left(\frac{r}{\alpha}\right)^a\frac{2-a}{2}t\dot{t} + r\dot{r} + \theta\dot{\theta}k^2 + z\dot{z}$
$\mathbf{Y_2}$	$\phi_6 = 2k^2[s\dot{\theta} - \theta]$
$\mathbf{Y_3}$	$\phi_7 = 2[s\dot{z} - z]$

Solution-II

Coefficients of the metric are

$$\nu(r) = c, \quad \mu(r) = b\ln\left(\frac{r}{\alpha}\right), \quad \lambda(r) = c_0.$$

Components of the Noether symmetry generators are

$$\xi = c_1 s + c_2, \quad \eta^0 = c_1\frac{t}{2} + c_3 s + c_4 z + c_5, \quad \eta^1 = c_1\frac{r}{2},$$
$$\eta^2 = c_1\frac{2-b}{4}\theta + c_6, \quad \eta^3 = c_1\frac{z}{2} + c_7 s + c_4 t + c_8, \quad A = 2c_3 t - 2c_7 z + c_9.$$

The corresponding spacetime is

$$ds^2 = dt^2 - dr^2 - \left(\frac{r}{\alpha}\right)^b k^2 d\theta^2 - dz^2, \quad b \neq 0, 2, \quad \alpha \neq 0. \tag{5.6.3}$$

The additional Noether symmetry generators are

$$\begin{aligned}&\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-b}{4}\theta\frac{\partial}{\partial \theta} + \frac{z}{2}\frac{\partial}{\partial z}, \quad \mathbf{Y_2} = s\frac{\partial}{\partial t}, \quad A_2 = 2t,\\ &\mathbf{Y_3} = s\frac{\partial}{\partial z}, \quad A_3 = -2z, \quad \mathbf{X_3} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}.\end{aligned} \tag{5.6.4}$$

The first integrals are given below in Table 5.10.

Table 5.10: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2[t\dot{z} - z\dot{t}]$
$\mathbf{Y_1}$	$\phi_5 = s\mathcal{L} - t\dot{t} + r\dot{r} + \left(\frac{r}{\alpha}\right)^b \frac{2-b}{2}\theta\dot{\theta}k^2 + z\dot{z}$
$\mathbf{Y_2}$	$\phi_6 = 2[t - s\dot{t}]$
$\mathbf{Y_3}$	$\phi_7 = 2[s\dot{z} - z]$

Solution-III

The metric coefficients are

$$\nu(r) = c, \quad \mu(r) = c_0, \quad \lambda(r) = c\ln\left(\frac{r}{\alpha}\right).$$

Components of Noether symmetry generators are

$$\xi = c_1 s + c_2, \quad \eta^0 = c_1\frac{t}{2} + c_3 s + c_4 k^2\theta + c_5, \quad \eta^1 = c_1\frac{r}{2},$$
$$\eta^2 = c_1\frac{\theta}{2} + c_4 t + c_6 s + c_7, \quad \eta^3 = c_1\frac{2-c}{4}z + c_8, \quad A = 2c_3 t - 2c_6 k^2\theta + c_9.$$

The corresponding spacetime takes the form

$$ds^2 = dt^2 - dr^2 - k^2 d\theta^2 - \left(\frac{r}{\alpha}\right)^c dz^2, \quad c \neq 0, 2, \quad \alpha \neq 0. \tag{5.6.5}$$

The additional Noether symmetry generators are

$$\begin{aligned}
&\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{\theta}{2}\frac{\partial}{\partial \theta} + \frac{2-c}{4}z\frac{\partial}{\partial z}, \quad \mathbf{Y_2} = s\frac{\partial}{\partial t}, \quad A_2 = 2t, \\
&\mathbf{Y_3} = s\frac{\partial}{\partial \theta}, \quad A_3 = -2k^2\theta, \quad \mathbf{X_3} = k^2\theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta}.
\end{aligned} \tag{5.6.6}$$

The first integral for these symmetries are given in the following Table 5.11.

Table 5.11: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2k^2[t\dot{\theta} - \theta\dot{t}]$
$\mathbf{Y_1}$	$\phi_5 = s\mathcal{L} - t\dot{t} + r\dot{r} + \theta\dot{\theta}k^2 + \frac{2-c}{2}z\dot{z}\left(\frac{r}{\alpha}\right)^c$
$\mathbf{Y_2}$	$\phi_6 = 2[t - s\dot{t}]$
$\mathbf{Y_3}$	$\phi_7 = 2k^2[s\dot{\theta} - \theta]$

Solution-IV:

Coefficients of the metric are

$$\nu(r) = c, \quad \mu(r) = 2\ln\left(\frac{r}{\alpha}\right), \quad \lambda(r) = 2\ln\left(\frac{r}{\alpha}\right).$$

The values of ξ, $\eta^j, j = 0,1,2,3$ and A are

$$\begin{aligned}
&\xi = c_1 s^2 + c_2 s + c_3, \quad \eta^0 = c_1\frac{ts}{2} + c_2\frac{t}{2} + c_6 s + c_5, \quad \eta^1 = c_1\frac{rs}{2} + c_2\frac{r}{2}, \\
&\eta^2 = -c_4 z + c_7, \quad \eta^3 = c_4 k^2\theta + c_8, \quad A = c_1\frac{(t^2 - r^2)}{2} + 2c_6 t + c_9.
\end{aligned}$$

The metric in this case is

$$ds^2 = dt^2 - dr^2 - \left(\frac{r}{\alpha}\right)^2 k^2 d\theta^2 - \left(\frac{r}{\alpha}\right)^2 dz^2, \quad \alpha \neq 0. \tag{5.6.7}$$

Here we have the following additional Noether symmetry generators

$$\begin{aligned}
&\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r}, \quad \mathbf{Y_2} = s^2\frac{\partial}{\partial s} + st\frac{\partial}{\partial t} + sr\frac{\partial}{\partial r}, \quad A_2 = t^2 - r^2, \\
&\mathbf{Y_3} = s\frac{\partial}{\partial t}, \quad A_3 = 2t, \quad \mathbf{X_3} = -z\frac{\partial}{\partial \theta} + k^2\theta\frac{\partial}{\partial z}.
\end{aligned} \tag{5.6.8}$$

The first integrals are given here in Table 5.12.

Table 5.12: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\left(\frac{r}{\alpha}\right)^2 k^2[\theta\dot{z} - z\dot{\theta}]$
$\mathbf{Y_1}$	$\phi_5 = s\mathcal{L} + r\dot{r} - t\dot{t}$
$\mathbf{Y_2}$	$\phi_6 = s^2\mathcal{L} + 2sr\dot{r} - 2st\dot{t} + t^2 - r^2$
$\mathbf{Y_3}$	$\phi_7 = 2[t - s\dot{t}]$

Solution-V:

The metric coefficients are

$$\nu(r) = 2\ln\left(\frac{r}{\alpha}\right), \quad \mu(r) = c, \quad \lambda(r) = 2\ln\left(\frac{r}{\alpha}\right).$$

Components of the Noether symmetry generators are

$$\xi = c_1s^2 + c_2s + c_3, \quad \eta^0 = c_4z + c_5, \quad \eta^1 = c_1\frac{rs}{2} + c_2\frac{r}{2},$$
$$\eta^2 = c_1\frac{s\theta}{2} + c_2\frac{\theta}{2} + c_8s + c_6, \quad \eta^3 = c_4t + c_7, \quad A = c_1\frac{(-r^2 - k^2\theta^2)}{2} - 2c_8k^2\theta + c_9.$$

The spacetime for this solution takes the form

$$ds^2 = \left(\frac{r}{\alpha}\right)^2 dt^2 - dr^2 - k^2d\theta^2 - \left(\frac{r}{\alpha}\right)^2 dz^2, \quad \alpha \neq 0. \tag{5.6.9}$$

The following four are the additional symmetries

$$\begin{aligned} &\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{\theta}{2}\frac{\partial}{\partial\theta}, \quad \mathbf{Y_2} = s^2\frac{\partial}{\partial s} + sr\frac{\partial}{\partial r} + s\theta\frac{\partial}{\partial\theta}, \quad A_2 = -r^2 - k^2\theta^2, \\ &\mathbf{Y_3} = s\frac{\partial}{\partial\theta}, \quad A_3 = -2k^2\theta, \quad \mathbf{X_3} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}. \end{aligned} \tag{5.6.10}$$

Table 5.13 contains the invariants of these symmetries

Table 5.13: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\left(\frac{r}{\alpha}\right)^2[t\dot{z} - z\dot{t}]$
$\mathbf{Y_1}$	$\phi_5 = s\mathcal{L} + r\dot{r} + \theta\dot{\theta}k^2$
$\mathbf{Y_2}$	$\phi_6 = s^2\mathcal{L} + 2sr\dot{r} + 2s\theta\dot{\theta}k^2 - r^2 - k^2\theta^2$
$\mathbf{Y_3}$	$\phi_7 = 2k^2[s\dot{\theta} - \theta]$

Solution-VI:

Coefficients of the spacetime are

$$\nu(r) = 2\ln\left(\frac{r}{\alpha}\right), \quad \mu(r) = 2\ln\left(\frac{r}{\alpha}\right), \quad \lambda(r) = c.$$

Components of the Noether symmetry generators are

$$\xi = c_1 s^2 + c_2 s + c_3, \quad \eta^0 = c_4 k^2 \theta + c_5, \quad \eta^1 = c_1 \frac{rs}{2} + c_2 \frac{r}{2},$$
$$\eta^2 = c_4 t + c_6, \quad \eta^3 = c_1 \frac{zs}{2} + c_2 \frac{z}{2} + c_8 s + c_7, \quad A = c_1 \frac{(-z^2 - r^2)}{2} - 2c_8 z + c_9.$$

The corresponding spacetime is

$$ds^2 = \left(\frac{r}{\alpha}\right)^2 dt^2 - dr^2 - \left(\frac{r}{\alpha}\right)^2 k^2 d\theta^2 - dz^2, \quad \alpha \neq 0. \tag{5.6.11}$$

The additional symmetries are

$$\begin{aligned} &\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{z}{2}\frac{\partial}{\partial z}, \quad \mathbf{Y_2} = s^2\frac{\partial}{\partial s} + sr\frac{\partial}{\partial t} + sz\frac{\partial}{\partial z}, \quad A_2 = -r^2 - z^2, \\ &\mathbf{Y_3} = s\frac{\partial}{\partial z}, \quad A_3 = -2z, \quad \mathbf{X_3} = k^2\theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta}. \end{aligned} \tag{5.6.12}$$

The first integrals are given in Table 5.14.

Table 5.14: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\left(\frac{r}{\alpha}\right)^2 k^2 [t\dot{\theta} - \theta\dot{t}]$
$\mathbf{Y_1}$	$\phi_6 = s\mathcal{L} + r\dot{r} + z\dot{z}$
$\mathbf{Y_2}$	$\phi_6 = s^2\mathcal{L} + 2sr\dot{r} + 2sz\dot{z} - r^2 - z^2$
$\mathbf{Y_3}$	$\phi_7 = 2[s\dot{z} - z]$

Solution-VII:

Coefficients of the spacetime are

$$\nu(r) = a\ln\left(\frac{r}{\alpha}\right), \quad \mu(r) = a\ln\left(\frac{r}{\alpha}\right), \quad \lambda(r) = a\ln\left(\frac{r}{\alpha}\right).$$

Components of the symmetry generators are

$$\xi = c_1 s + c_2, \quad \eta^0 = c_1 \frac{2-a}{4} t + c_3 k^2 \theta + c_4 z + c_5, \quad \eta^1 = c_1 \frac{r}{2},$$
$$\eta^2 = c_1 \frac{2-a}{4}\theta + c_3 t + c_6 z + c_7, \quad \eta^3 = c_1 \frac{2-a}{4} z + c_4 t - c_6 k^2 \theta + c_8, \quad A = c_9.$$

The spacetime is

$$ds^2 = \left(\frac{r}{\alpha}\right)^a dt^2 - dr^2 - \left(\frac{r}{\alpha}\right)^a k^2 d\theta^2 - \left(\frac{r}{\alpha}\right)^a dz^2, \quad k \neq 0, 2, \quad \alpha \neq 0. \tag{5.6.13}$$

The symmetries other than the minimal set are

$$\begin{aligned} \mathbf{Y_1} &= s\frac{\partial}{\partial s} + \frac{2-a}{4}t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-a}{4}\theta\frac{\partial}{\partial \theta} + \frac{2-a}{4}z\frac{\partial}{\partial z}, \\ \mathbf{X_3} &= z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \quad \mathbf{X_4} = k^2\theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta}, \quad \mathbf{X_5} = z\frac{\partial}{\partial \theta} - k^2\theta\frac{\partial}{\partial z}. \end{aligned} \tag{5.6.14}$$

The first integrals are given in Table 5.15.

Table 5.15: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\left(\frac{r}{\alpha}\right)^a[t\dot{z} - z\dot{t}]$
$\mathbf{X_4}$	$\phi_5 = 2\left(\frac{r}{\alpha}\right)^a k^2[t\dot{\theta} - \theta\dot{t}]$
$\mathbf{X_5}$	$\phi_6 = 2\left(\frac{r}{\alpha}\right)^a k^2[z\dot{\theta} - \theta\dot{z}]$
$\mathbf{Y_1}$	$\phi_7 = s\mathcal{L} + \left(\frac{r}{\alpha}\right)^a \frac{(2-a)}{2}[-t\dot{t} + \theta\dot{\theta}k^2 + z\dot{z}] + r\dot{r}$

5.7 Nine Noether Symmetries and First Integrals

The classes for nine Noether symmetries are given in this section. There are four classes of the cylindrically symmetric static spacetimes the actions of the Lagrangians of which admit nine Noether symmetries. The detail discussion on these spacetime is given below:

Solution-I:

The metric coefficients are

$$\nu(r) = c, \quad \mu(r) = \frac{r}{\alpha} = \lambda(r).$$

Components of the Noether symmetry generators are

$$\begin{aligned} &\xi = c_1, \quad \eta^0 = c_4 s + c_5, \quad \eta^1 = c_2 + c_6 z + c_7, \quad A = 2c_4 t + c_{10}, \\ &\eta^2 = -c_2\frac{k^2\theta}{2\alpha} - c_3 z - c_6\frac{z\theta}{2\alpha} + c_7\frac{(-k^2\theta^2 + z^2 + 4\alpha^2 e^{\frac{-r}{\alpha}})}{4\alpha} + c_8, \\ &\eta^3 = -c_2\frac{z}{2\alpha} + c_3 k^2\theta - c_7\frac{zk^2\theta}{2\alpha} + c_6\frac{(k^2\theta^2 - z^2 + 4\alpha^2 e^{\frac{-r}{\alpha}})}{4\alpha} + c_9. \end{aligned}$$

The spacetime takes the form

$$ds^2 = dt^2 - dr^2 - e^{\frac{r}{\alpha}}k^2 d\theta^2 - e^{\frac{r}{\alpha}}dz^2, \quad \alpha \neq 0. \tag{5.7.1}$$

The Noether symmetry generators other than the minimal set are

$$\begin{aligned}
\mathbf{X_3} &= \frac{\partial}{\partial r} - \frac{\theta}{2\alpha}\frac{\partial}{\partial\theta} - \frac{z}{2\alpha}\frac{\partial}{\partial z}, \mathbf{X_4} = -z\frac{\partial}{\partial\theta} + k^2\theta\frac{\partial}{\partial z},\\
\mathbf{X_5} &= z\frac{\partial}{\partial r} - \frac{z\theta}{2\alpha}\frac{\partial}{\partial\theta} + [\frac{k^2\theta^2 - z^2 + 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial z},\\
\mathbf{X_6} &= k^2\theta\frac{\partial}{\partial r} + [\frac{-k^2\theta^2 + z^2 + 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial\theta} - \frac{k^2 z\theta}{2\alpha}\frac{\partial}{\partial z},\\
\mathbf{Y_1} &= s\frac{\partial}{\partial t}, \qquad A = 2t.
\end{aligned} \tag{5.7.2}$$

Table 5.16 contains the corresponding first integrals.

Table 5.16: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = -\frac{k^2\theta\dot{\theta}e^{\frac{r}{\alpha}}}{\alpha} + 2\dot{r} - \frac{z\dot{z}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_4}$	$\phi_5 = 2e^{\frac{r}{\alpha}}k^2[\theta\dot{z} - z\dot{\theta}]$
$\mathbf{X_5}$	$\phi_6 = \frac{[(k^2\theta^2 - z^2)e^{\frac{r}{\alpha}} + 4\alpha^2]\dot{z}}{2\alpha} + 2z\dot{r} - \frac{\theta z\dot{\theta}k^2 e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_6}$	$\phi_7 = \frac{[(-k^2\theta^2 + z^2)e^{\frac{r}{\alpha}} + 4\alpha^2]\dot{\theta}}{2\alpha} + 2\theta\dot{r} - \frac{\theta z\dot{z}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{Y_1}$	$\phi_8 = 2[-s\dot{t} + t]$

Solution-II:

Coefficients of the metrics are

$$\nu(r) = \frac{r}{\alpha}, \quad \mu(r) = c, \quad \lambda(r) = \frac{r}{\alpha}.$$

Components of the symmetry generators are

$$\begin{aligned}
&\xi = c_1, \quad \eta^0 = -c_2\frac{t}{2\alpha} + c_3 z + c_4\frac{-t^2 + z^2 + 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha} - c_5\frac{tz}{2\alpha} + c_6,\\
&\eta^1 = c_2 + c_4 t + c_5 z, \quad \eta^2 = c_7 s + c_8, \quad A = -2c_7 k^2\theta + c_{10},\\
&\eta^0 = -c_2\frac{z}{2\alpha} + c_3 t + c_5\frac{-t^2 - z^2 + 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha} - c_4\frac{tz}{2\alpha} + c_9.
\end{aligned}$$

The metric in this case is

$$ds^2 = e^{\frac{r}{\alpha}}dt^2 - dr^2 - k^2 d\theta^2 - e^{\frac{r}{\alpha}}dz^2, \quad \alpha \neq 0. \tag{5.7.3}$$

The symmetries other than the minimal set are

$$\begin{aligned}
\mathbf{X_3} &= \frac{\partial}{\partial r} - \frac{t}{2\alpha}\frac{\partial}{\partial t} - \frac{z}{2\alpha}\frac{\partial}{\partial z}, \quad \mathbf{X_4} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \\
\mathbf{X_5} &= t\frac{\partial}{\partial r} - [\frac{t^2 - z^2 + 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial t} - \frac{zt}{2\alpha}\frac{\partial}{\partial z}, \\
\mathbf{X_6} &= z\frac{\partial}{\partial r} - \frac{zt}{2\alpha}\frac{\partial}{\partial t} - [\frac{t^2 + z^2 - 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial z}, \\
\mathbf{Y_1} &= s\frac{\partial}{\partial \theta}, \qquad A = -2k^2\theta.
\end{aligned} \tag{5.7.4}$$

The corresponding first integrals are given in Table 5.17.

Table 5.17: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = \frac{t\dot{t}e^{\frac{r}{\alpha}}}{\alpha} + 2\dot{r} - \frac{z\dot{z}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_4}$	$\phi_5 = 2e^{\frac{r}{\alpha}}[t\dot{z} - z\dot{t}]$
$\mathbf{X_5}$	$\phi_6 = \frac{[(t^2-z^2)e^{\frac{r}{\alpha}}+4\alpha^2]\dot{t}}{2\alpha} + 2t\dot{r} - \frac{tz\dot{z}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_6}$	$\phi_7 = -\frac{[(t^2+z^2)e^{\frac{r}{\alpha}}-4\alpha^2]\dot{z}}{2\alpha} + 2z\dot{r} - \frac{tz\dot{t}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{Y_1}$	$\phi_8 = 2k^2[s\dot{\theta} - \theta]$

Solution-III:

The metrics coefficients are

$$\nu(r) = \frac{r}{\alpha}, \quad \mu(r) = \frac{r}{\alpha}, \quad \lambda(r) = c.$$

Components of the Noether symmetry generators are

$$\begin{aligned}
&\xi = c_1, \quad \eta^0 = -c_2\frac{t}{2\alpha} + c_3k^2\theta + c_4\frac{-t^2 + k^2\theta^2 + 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha} - c_5\frac{k^2 t\theta}{2\alpha} + c_6, \\
&\eta^1 = c_2 + c_4 t + c_5 k^2\theta, \quad \eta^2 = -c_2\frac{\theta}{2\alpha} + c_3 t + c_5\frac{-t^2 + k^2\theta^2 + 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha} - c_4\frac{t\theta}{2\alpha} + c_7, \\
&\eta^3 = c_8 s + c_9, \quad A = -2c_8 z + c_{10}.
\end{aligned}$$

The metric takes the form

$$ds^2 = e^{\frac{r}{\alpha}}dt^2 - dr^2 - e^{\frac{r}{\alpha}}k^2 d\theta^2 - dz^2, \quad \alpha \neq 0. \tag{5.7.5}$$

The symmetries other than the minimal set are

$$\begin{aligned}
&\mathbf{X_3} = \frac{\partial}{\partial r} - \frac{t}{2\alpha}\frac{\partial}{\partial t} - \frac{\theta}{2\alpha}\frac{\partial}{\partial \theta}, \quad \mathbf{X_4} = k^2\theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta}, \\
&\mathbf{X_5} = t\frac{\partial}{\partial r} - [\frac{t^2 - k^2\theta^2 + 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial t} - \frac{t\theta}{2\alpha}\frac{\partial}{\partial \theta}, \\
&\mathbf{X_6} = k^2\theta\frac{\partial}{\partial r} - \frac{k^2 t\theta}{2\alpha}\frac{\partial}{\partial t} - [\frac{t^2 - k^2\theta^2 - 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial \theta}, \\
&\mathbf{Y_1} = s\frac{\partial}{\partial z}, \qquad A = -2z.
\end{aligned} \tag{5.7.6}$$

The corresponding first integrals are given in Table 5.18.

Table 5.18: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = \frac{t\dot{t}e^{\frac{r}{\alpha}}}{\alpha} + 2\dot{r} - \frac{\theta\dot{\theta}e^{\frac{r}{\alpha}}k^2}{\alpha}$
$\mathbf{X_4}$	$\phi_5 = 2e^{\frac{r}{\alpha}}k^2[t\dot{\theta} - \theta\dot{t}]$
$\mathbf{X_5}$	$\phi_6 = \frac{[(t^2-k^2\theta^2)e^{\frac{r}{\alpha}}+4\alpha^2]\dot{t}}{2\alpha} + 2t\dot{r} - \frac{t\theta\dot{\theta}k^2e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_6}$	$\phi_7 = -\frac{[(t^2-k^2\theta^2)e^{\frac{r}{\alpha}}-4\alpha^2]\dot{\theta}}{2\alpha} + 2\theta\dot{r} + \frac{t\theta\dot{t}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{Y_1}$	$\phi_8 = 2[s\dot{z} - z]$

Solution-IV:

Coefficients of the metric are

$$\nu(r) = \mu(r) = \lambda(r) = 2\ln\left(\frac{r}{\alpha}\right).$$

Components of the symmetry generators are

$$\xi = c_1 s^2 + c_2 s + c_3, \quad \eta^0 = c_4 z + c_6 k^2\theta + c_7, \quad \eta^1 = c_1 s r + c_2\frac{r}{\alpha},$$
$$\eta^2 = c_5 z + c_6 t + c_8, \quad \eta^3 = c_4 t - c_5 k^2\theta + c_9, \quad A = -c_1 r^2 + c_{10}.$$

The spacetime takes the form

$$ds^2 = \left(\frac{r}{\alpha}\right)^2 dt^2 - dr^2 - \left(\frac{r}{\alpha}\right)^2 k^2 d\theta^2 - \left(\frac{r}{\alpha}\right)^2 dz^2, \quad \alpha \neq 0. \tag{5.7.7}$$

The following five are the additional symmetries

$$\begin{aligned}
&\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{r}{2}\frac{\partial}{\partial r}, \qquad \mathbf{Y_2} = s^2\frac{\partial}{\partial s} + sr\frac{\partial}{\partial t}, \qquad A_2 = -r^2, \\
&\mathbf{X_3} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \qquad \mathbf{X_4} = z\frac{\partial}{\partial \theta} - k^2\theta\frac{\partial}{\partial z}, \qquad \mathbf{X_5} = k^2\theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta}.
\end{aligned} \tag{5.7.8}$$

The corresponding first integrals or conservation laws are given in Table 5.19.

Table 5.19: First Integrals

Gen	First Integrals
$\mathbf{Y_1}$	$\phi_4 = s\mathcal{L} + r\dot{r}$
$\mathbf{Y_2}$	$\phi_5 = s^2\mathcal{L} + 2sr\dot{r} - r^2$
$\mathbf{X_3}$	$\phi_6 = 2\left(\frac{r}{\alpha}\right)^2[t\dot{z} - z\dot{t}]$
$\mathbf{X_4}$	$\phi_7 = 2\left(\frac{r}{\alpha}\right)^2 k^2[z\dot{\theta} - \theta\dot{z}]$
$\mathbf{X_5}$	$\phi_8 = 2\left(\frac{r}{\alpha}\right)^2 k^2[t\dot{\theta} - \theta\dot{t}]$

5.8 Eleven Noether symmetries and First Integrals

There is only one class of cylindrically symmetric static spacetime the action of the Lagrangian of which admits eleven Noether symmetries, the detail is given as follows:

Solution:

Coefficients of the metric for eleven Noether symmetries are

$$\nu(r) = \mu(r) = \lambda(r) = \frac{r}{\alpha}.$$

Components of the Noether symmetry generators are

$$\eta^0 = -c_2\frac{t}{2\alpha} + c_3k^2\theta + c_4z - c_6\frac{t^2 + k^2\theta^2 + z^2 + 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha} + c_7\frac{tz}{2\alpha} + c_8\frac{k^2t\theta}{2\alpha} + c_9,$$

$$\eta^1 = c_2 + c_6t + c_7z + c_8k^2\theta, \quad \xi = c_1, \quad A = c_{12},$$

$$\eta^2 = -c_2\frac{\theta}{2\alpha} + c_3t + c_5z - c_8\frac{t^2 + k^2\theta^2 - z^2 - 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha} - c_6\frac{t\theta}{2\alpha} - c_7\frac{z\theta}{2\alpha} + c_{10},$$

$$\eta^3 = -c_2\frac{z}{2\alpha} + c_4t - c_5\theta - c_7\frac{t^2 - k^2\theta^2 + z^2 - 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha} - c_6\frac{tz}{2\alpha} - c_8\frac{k^2z\theta}{2\alpha} + c_{11}.$$

The spacetime takes the form

$$ds^2 = e^{\frac{r}{\alpha}}dt^2 - dr^2 - e^{\frac{r}{\alpha}}k^2d\theta^2 - e^{\frac{r}{\alpha}}dz^2, \quad \alpha \neq 0. \tag{5.8.1}$$

We get the following seven Noether symmetries along with the minimal set

$$\begin{aligned}
&\mathbf{X_3} = \frac{\partial}{\partial r} - \frac{t}{2\alpha}\frac{\partial}{\partial t} - \frac{\theta}{2\alpha}\frac{\partial}{\partial \theta} - \frac{z}{2\alpha}\frac{\partial}{\partial z}, \qquad \mathbf{X_4} = k^2\theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta}, \\
&\mathbf{X_5} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \qquad \mathbf{X_6} = z\frac{\partial}{\partial \theta} - k^2\theta\frac{\partial}{\partial z}, \\
&\mathbf{X_7} = t\frac{\partial}{\partial r} - [\frac{t^2 + k^2\theta^2 + z^2 + 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial t} - \frac{t\theta}{2\alpha}\frac{\partial}{\partial \theta} - \frac{tz}{2\alpha}\frac{\partial}{\partial z}, \\
&\mathbf{X_8} = z\frac{\partial}{\partial r} + \frac{tz}{2\alpha}\frac{\partial}{\partial t} - \frac{\theta z}{2\alpha}\frac{\partial}{\partial \theta} + [\frac{-t^2 + k^2\theta^2 - z^2 + 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial z}, \\
&\mathbf{X_9} = k^2\theta\frac{\partial}{\partial r} + \frac{k^2 t\theta}{2\alpha}\frac{\partial}{\partial t} + [\frac{-t^2 - k^2\theta^2 + z^2 + 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial \theta} - \frac{k^2\theta z}{2\alpha}\frac{\partial}{\partial z}.
\end{aligned} \tag{5.8.2}$$

The first integrals are given in Table 5.20.

Table 5.20: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\dot{r} + \frac{t\dot{t}e^{\frac{r}{\alpha}}}{\alpha} - \frac{\theta\dot{\theta}e^{\frac{r}{\alpha}}k^2}{\alpha} - \frac{z\dot{z}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_4}$	$\phi_5 = 2e^{\frac{r}{\alpha}}k^2[t\dot{\theta} - \theta\dot{t}]$
$\mathbf{X_5}$	$\phi_6 = 2e^{\frac{r}{\alpha}}[t\dot{z} - z\dot{t}]$
$\mathbf{X_6}$	$\phi_7 = 2e^{\frac{r}{\alpha}}k^2[z\dot{\theta} - \theta\dot{z}]$
$\mathbf{X_7}$	$\phi_8 = \frac{(t^2+k^2\theta^2+z^2)e^{\frac{r}{\alpha}}+4\alpha^2}{2\alpha}\dot{t} + 2t\dot{r} - \frac{\theta t\dot{\theta}e^{\frac{r}{\alpha}}k^2}{\alpha} - \frac{zt\dot{z}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_8}$	$\phi_9 = \frac{(-t^2+k^2\theta^2-z^2)e^{\frac{r}{\alpha}}+4\alpha^2}{2\alpha}\dot{z} + 2z\dot{r} - \frac{\theta z\dot{\theta}e^{\frac{r}{\alpha}}k^2}{\alpha} - \frac{zt\dot{t}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_9}$	$\phi_{10} = \frac{(-t^2-k^2\theta^2+z^2)e^{\frac{r}{\alpha}}+4\alpha^2}{2\alpha}\dot{\theta} + 2\theta\dot{r} - \frac{\theta t\dot{t}e^{\frac{r}{\alpha}}}{\alpha} - \frac{z\theta\dot{z}e^{\frac{r}{\alpha}}}{\alpha}$

5.9 Seventeen Noether Symmetries and First Integrals

There are four classes for seventeen Noether symmetries:

Solution-I:

The metric coefficients are

$$\nu(r) = 2\ln\left(\frac{r}{\alpha}\right), \quad \mu(r) = a, \quad \lambda(r) = b.$$

Components of the symmetry generators are

$$\eta^0 = -c_4\frac{z\alpha e^{\frac{-t}{\alpha}}}{r} + c_5\frac{z\alpha e^{\frac{t}{\alpha}}}{r} + c_6\frac{k^2\theta\alpha e^{\frac{-t}{\alpha}}}{r} - c_8\frac{\alpha e^{\frac{-t}{\alpha}}}{r} + c_9\frac{\alpha e^{\frac{t}{\alpha}}}{r} + c_{10}\frac{k^2\theta\alpha e^{\frac{t}{\alpha}}}{r} - c_{12}\frac{se^{\frac{-t}{\alpha}}}{r} + c_{13}\frac{se^{\frac{t}{\alpha}}}{r} + c_2\frac{t}{2} + c_{15},$$

$$\eta^1 = -c_4 ze^{\frac{-t}{\alpha}} - c_5 ze^{\frac{t}{\alpha}} - c_6 k^2\theta e^{\frac{-t}{\alpha}} + c_8 e^{\frac{-t}{\alpha}} - c_9 e^{\frac{t}{\alpha}} - c_{10}k^2\theta e^{\frac{t}{\alpha}} - c_{12}se^{\frac{-t}{\alpha}} - c_{13}se^{\frac{t}{\alpha}} + c_2\frac{r}{2},$$

$$\eta^2 = c_6 re^{\frac{-t}{\alpha}} + c_7 z + c_{10}re^{\frac{t}{\alpha}} + c_2\frac{\theta}{2} + c_{11}s + c_1 s\theta + c_{16},$$

$$\eta^3 = c_4 re^{\frac{-t}{\alpha}} - c_7 k^2\theta + c_5 re^{\frac{t}{\alpha}} + c_2\frac{z}{2} + c_{14}s + c_1 sz + c_{17},$$

$$A = -2c_{11}k^2\theta - c_1(r^2 + k^2\theta^2 + z^2) + 2c_{12}re^{\frac{-t}{\alpha}} + 2c_{13}re^{\frac{t}{\alpha}} - 2c_{14}z + c_{18},$$

$$\xi = c_1 s^2 + c_2 s + c_3.$$

The metric takes the form

$$ds^2 = \left(\frac{r}{\alpha}\right)^2 dt^2 - dr^2 - k^2 d\theta^2 - dz^2, \quad \alpha \neq 0. \tag{5.9.1}$$

The additional symmetries are

$$\begin{aligned}
&\mathbf{X_3} = -\frac{ze^{\frac{-t}{\alpha}}\alpha}{r}\frac{\partial}{\partial t} - ze^{\frac{-t}{\alpha}}\frac{\partial}{\partial r} + re^{\frac{-t}{\alpha}}\frac{\partial}{\partial z}, \qquad \mathbf{X_4} = \frac{ze^{\frac{t}{\alpha}}\alpha}{r}\frac{\partial}{\partial t} - ze^{\frac{t}{\alpha}}\frac{\partial}{\partial r} + re^{\frac{t}{\alpha}}\frac{\partial}{\partial z},\\
&\mathbf{X_5} = \frac{\theta e^{\frac{-t}{\alpha}}\alpha}{r}\frac{\partial}{\partial t} - \theta e^{\frac{-t}{\alpha}}\frac{\partial}{\partial r} + \frac{re^{\frac{-t}{\alpha}}}{k^2}\frac{\partial}{\partial \theta}, \qquad \mathbf{X_6} = z\frac{\partial}{\partial \theta} - k^2\theta\frac{\partial}{\partial z},\\
&\mathbf{X_7} = -\frac{e^{\frac{-t}{\alpha}}\alpha}{r}\frac{\partial}{\partial t} + e^{\frac{-t}{\alpha}}\frac{\partial}{\partial r}, \qquad \mathbf{X_8} = \frac{e^{\frac{t}{\alpha}}\alpha}{r}\frac{\partial}{\partial t} - e^{\frac{t}{\alpha}}\frac{\partial}{\partial r},\\
&\mathbf{X_9} = \frac{\theta e^{\frac{t}{\alpha}}\alpha}{r}\frac{\partial}{\partial t} - \theta e^{\frac{t}{\alpha}}\frac{\partial}{\partial r} + \frac{re^{\frac{t}{\alpha}}}{k^2}\frac{\partial}{\partial \theta},\\
&\mathbf{Y_1} = 2s\frac{\partial}{\partial s} + r\frac{\partial}{\partial r} + \theta\frac{\partial}{\partial \theta} + z\frac{\partial}{\partial z}, \qquad \mathbf{Y_2} = s\frac{\partial}{\partial \theta}, \qquad A_2 = -2k^2\theta,\\
&\mathbf{Y_3} = s[s\frac{\partial}{\partial s} + r\frac{\partial}{\partial r} + \theta\frac{\partial}{\partial \theta} + z\frac{\partial}{\partial z}], \qquad A_3 = -r^2 - k^2\theta^2 - z^2,\\
&\mathbf{Y_4} = -s[\frac{e^{\frac{-t}{\alpha}}\alpha}{r}\frac{\partial}{\partial t} + e^{\frac{-t}{\alpha}}\frac{\partial}{\partial r}], \qquad A_4 = 2re^{\frac{-t}{\alpha}},\\
&\mathbf{Y_5} = s[\frac{e^{\frac{t}{\alpha}}\alpha}{r}\frac{\partial}{\partial t} - e^{\frac{t}{\alpha}}\frac{\partial}{\partial r}], \qquad A_5 = 2re^{\frac{t}{\alpha}}, \qquad \mathbf{Y_6} = s\frac{\partial}{\partial z}, \qquad A_6 = -2z.
\end{aligned} \tag{5.9.2}$$

Table 5.21 contains the invariants of these symmetries

Table 5.21: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2e^{\frac{-t}{\alpha}}[\frac{rz\dot{t}}{\alpha} - z\dot{r} + r\dot{z}]$
$\mathbf{X_4}$	$\phi_5 = 2e^{\frac{t}{\alpha}}[-\frac{rz\dot{t}}{\alpha} - z\dot{r} + r\dot{z}]$
$\mathbf{X_5}$	$\phi_6 = 2e^{\frac{-t}{\alpha}}[-\frac{r\theta\dot{t}}{\alpha} - \theta\dot{r} + r\dot{\theta}]$
$\mathbf{X_6}$	$\phi_7 = 2k^2[z\dot{\theta} - \theta\dot{z}]$
$\mathbf{X_7}$	$\phi_8 = 2e^{\frac{-t}{\alpha}}[\frac{\dot{t}r}{\alpha} + \dot{r}]$
$\mathbf{X_8}$	$\phi_9 = -2e^{\frac{t}{\alpha}}[\frac{\dot{t}r}{\alpha} + \dot{r}]$
$\mathbf{X_9}$	$\phi_{10} = 2e^{\frac{t}{\alpha}}[-\frac{r\theta\dot{t}}{\alpha} - \theta\dot{r} + r\dot{\theta}]$
$\mathbf{Y_1}$	$\phi_{11} = 2[s\mathcal{L} + r\dot{r} + k^2\theta\dot{\theta} + z\dot{z}]$
$\mathbf{Y_2}$	$\phi_{12} = 2k^2[s\dot{\theta} - \theta]$
$\mathbf{Y_3}$	$\phi_{13} = s^2\mathcal{L} + 2s[r\dot{r} + k^2\theta\dot{\theta} + z\dot{z}] - r^2 - k^2\theta^2 - z^2$
$\mathbf{Y_4}$	$\phi_{14} = 2se^{\frac{-t}{\alpha}}[\frac{\dot{t}r}{\alpha} - \dot{r}] + 2re^{\frac{-t}{\alpha}}$
$\mathbf{Y_5}$	$\phi_{15} = -2se^{\frac{t}{\alpha}}[\frac{\dot{t}r}{\alpha} + \dot{r}] + 2re^{\frac{t}{\alpha}}$
$\mathbf{Y_6}$	$\phi_{16} = 2[s\dot{z} - z]$

Solution-II:

The metric coefficients are

$$\nu(r) = a, \quad \mu(r) = 2\ln\left(\frac{r}{\alpha}\right), \quad \lambda(r) = b.$$

Components of the Noether symmetry generators are

$$\eta^0 = c_4 r\cos\theta + c_5 r\sin\theta + c_7 z + c_2\frac{t}{2} + c_{11}s + c_1 ts + c_{15},$$

$$\eta^1 = c_4 t\cos\theta + c_5 t\sin\theta - c_6 z\cos\theta - c_8 z\sin\theta + c_9\sin\theta + c_{10}\cos\theta + c_2\frac{r}{\alpha} +$$

$$c_1 rs - c_{12}s\cos\theta - c_{13}s\sin\theta, \quad \xi = c_1 s^2 + c_2 s + c_3$$

$$\eta^2 = -c_4\frac{t\alpha\sin\theta}{kr} + c_5\frac{t\alpha\cos\theta}{kr} + c_6\frac{z\alpha\sin\theta}{kr} - c_8\frac{z\alpha\cos\theta}{kr} - c_9\frac{\alpha\cos\theta}{kr} -$$

$$c_{10}\frac{\alpha\sin\theta}{kr} + c_{12}\frac{s\sin\theta}{kr} - c_{13}\frac{s\cos\theta}{kr} + c_{16},$$

$$\eta^3 = c_6 r\cos\theta + c_8 r\sin\theta + c_7 t + c_2\frac{z}{2} + c_1 zs + c_{14}s + c_{17},$$

$$A = 2c_{11}t + c_1(t^2 - r^2 - z^2) - 2c_{12}r\cos\theta - 2c_{13}r\cos\theta - 2c_{14}z + c_{18}.$$

The metric in this case is

$$ds^2 = dt^2 - dr^2 - \left(\frac{r}{\alpha}\right)^2 k^2 d\theta^2 - dz^2, \quad \alpha \neq 0. \tag{5.9.3}$$

We get the following 13 additional Noether symmetry generators

$$\begin{aligned}
&\mathbf{X_3} = r\cos\theta\frac{\partial}{\partial t} - \frac{\alpha t\sin\theta}{kr}\frac{\partial}{\partial\theta} + t\cos\theta\frac{\partial}{\partial r}, \qquad \mathbf{X_4} = r\sin\theta\frac{\partial}{\partial t} + \frac{\alpha t\cos\theta}{kr}\frac{\partial}{\partial\theta} + t\sin\theta\frac{\partial}{\partial r},\\
&\mathbf{X_5} = -z\cos\theta\frac{\partial}{\partial r} + \frac{\alpha z\sin\theta}{kr}\frac{\partial}{\partial\theta} + r\cos\theta\frac{\partial}{\partial z}, \qquad \mathbf{X_6} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z},\\
&\mathbf{X_7} = -z\sin\theta\frac{\partial}{\partial r} - \frac{\alpha z\cos\theta}{kr}\frac{\partial}{\partial\theta} + r\sin\theta\frac{\partial}{\partial z}, \qquad \mathbf{X_8} = \sin\theta\frac{\partial}{\partial r} - \frac{\alpha\cos\theta}{kr}\frac{\partial}{\partial\theta},\\
&\mathbf{X_9} = \cos\theta\frac{\partial}{\partial r} - \frac{\alpha\sin\theta}{kr}\frac{\partial}{\partial\theta}, \quad \mathbf{Y_1} = 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z},\\
&\mathbf{Y_2} = s\frac{\partial}{\partial t}, \qquad A_2 = 2t, \quad \mathbf{Y_3} = s[s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}], \qquad A_3 = t^2 - r^2 - z^2,\\
&\mathbf{Y_4} = s[-\cos\theta\frac{\partial}{\partial r} + \frac{\alpha\sin\theta}{kr}\frac{\partial}{\partial\theta}], \qquad A_4 = -2r\cos\frac{\theta}{r}, \quad \mathbf{Y_5} = -s[\sin\theta\frac{\partial}{\partial r} + \frac{\alpha\cos\theta}{kr}\frac{\partial}{\partial\theta}],\\
&A_5 = -2r\sin\frac{\theta}{kr}, \qquad \mathbf{Y_6} = s\frac{\partial}{\partial z}, \qquad A_6 = -2z.
\end{aligned} \tag{5.9.4}$$

First integrals are given in the following Table 5.22.

Table 5.22: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2[-r\dot{t}\cos\theta - \frac{\dot{\theta}ktr\sin\theta}{\alpha} + t\dot{r}\cos\theta]$
$\mathbf{X_4}$	$\phi_5 = 2[-r\dot{t}\sin\theta + \frac{\dot{\theta}ktr\cos\theta}{\alpha} + t\dot{r}\sin\theta]$
$\mathbf{X_5}$	$\phi_6 = 2[-z\dot{r}\cos\theta + \frac{\dot{\theta}kzr\sin\theta}{\alpha} + r\dot{\theta}\cos\theta]$
$\mathbf{X_6}$	$\phi_7 = 2[t\dot{z} - z\dot{t}]$
$\mathbf{X_7}$	$\phi_8 = 2[-z\dot{r}\sin\theta - \frac{\dot{\theta}krz\cos\theta}{\alpha} + r\dot{z}\sin\theta]$
$\mathbf{X_8}$	$\phi_9 = 2[\dot{r}\sin\theta - \dot{\theta}k\frac{r\cos\theta}{\alpha}]$
$\mathbf{X_9}$	$\phi_{10} = 2[\dot{r}\cos\theta - \dot{\theta}k\frac{r\sin\theta}{\alpha}]$
$\mathbf{Y_1}$	$\phi_{11} = 2[s\mathcal{L} - t\dot{t} + r\dot{r} + z\dot{z}]$
$\mathbf{Y_2}$	$\phi_{12} = 2[t - s\dot{t}]$
$\mathbf{Y_3}$	$\phi_{13} = s^2\mathcal{L} - 2s[t\dot{t} - r\dot{r} - z\dot{z}] + t^2 - r^2 - z^2$
$\mathbf{Y_4}$	$\phi_{14} = 2s[-\dot{r}\cos\theta + \frac{\dot{\theta}kr\sin\theta}{\alpha}] - 2r\cos\theta$
$\mathbf{Y_5}$	$\phi_{15} = -2s[\dot{r}\sin\theta + \frac{\dot{\theta}kr\cos\theta}{\alpha}] + 2r\sin\theta$
$\mathbf{Y_6}$	$\phi_{16} = 2[s\dot{z} - z]$

Solution-III:

The metric coefficients for seventeen Noether symmetries are

$$\nu(r) = a, \quad \mu(r) = b, \quad \lambda(r) = 2\ln\left(\frac{r}{\alpha}\right).$$

The values of ξ, $\eta^j, j = 0, 1, 2, 3$ and A are

$$\eta^0 = c_4 r\cos(\frac{z}{\alpha}) + c_5 r\sin(\frac{z}{\alpha}) + c_7 k^2\theta + c_2\frac{t}{2} + c_{11}s + c_1 ts + c_{15},$$
$$\eta^1 = c_4 t\cos(\frac{z}{\alpha}) + c_5 t\sin(\frac{z}{\alpha}) - c_6 k^2\theta\cos(\frac{z}{\alpha}) - c_8 k^2\theta\sin(\frac{z}{\alpha}) + c_9\sin(\frac{z}{\alpha}) + c_{10}\cos(\frac{z}{\alpha}) + c_2\frac{r}{\alpha} +$$
$$c_1 rs - c_{12}s\cos(\frac{z}{\alpha}) - c_{13}\sin(\frac{z}{\alpha}), \quad \xi = c_1 s^2 + c_2 s + c_3.$$
$$\eta^2 = c_6 r\cos(\frac{z}{\alpha}) + c_8 r\sin(\frac{z}{\alpha}) + c_7 t + c_2\frac{\theta}{2} + c_1\theta s + c_{14}s + c_{16}$$
$$\eta^3 = -c_4\frac{t\alpha\sin(\frac{z}{\alpha})}{r} + c_5\frac{t\alpha\cos(\frac{z}{\alpha})}{r} + c_6\frac{k^2\theta\alpha\sin(\frac{z}{\alpha})}{r} - c_8\frac{k^2\theta\alpha\cos(\frac{z}{\alpha})}{r} - c_9\frac{\alpha\cos(\frac{z}{\alpha})}{r} -$$
$$c_{10}\frac{\alpha\sin(\frac{z}{\alpha})}{r} + c_{12}\frac{s\sin(\frac{z}{\alpha})}{r} - c_{13}\frac{s\alpha\cos(\frac{z}{\alpha})}{r} + c_{17},$$
$$A = 2c_{11}t + c_1(t^2 - r^2 - k^2\theta^2) + 2c_{12}r\cos(\frac{z}{r}) + 2c_{13}r\sin(\frac{z}{r}) - 2c_{14}\theta + c_{18}.$$

The spacetime takes the form

$$ds^2 = dt^2 - dr^2 - k^2 d\theta^2 - \left(\frac{r}{\alpha}\right)^2 dz^2, \quad \alpha \neq 0. \tag{5.9.5}$$

The action of this spacetime have the following 13 additional Noether symmetry generators other than the minimal set.

$$\begin{aligned}
&\mathbf{X_3} = r\cos\frac{z}{\alpha}\frac{\partial}{\partial t} - \frac{\alpha t\sin\frac{z}{\alpha}}{r}\frac{\partial}{\partial z} + t\cos\frac{z}{\alpha}\frac{\partial}{\partial r}, \qquad \mathbf{X_4} = r\sin\frac{z}{\alpha}\frac{\partial}{\partial t} + \frac{\alpha t\cos\frac{z}{\alpha}}{r}\frac{\partial}{\partial z} + t\sin\frac{z}{\alpha}\frac{\partial}{\partial r},\\
&\mathbf{X_5} = -k^2\theta\cos\frac{z}{\alpha}\frac{\partial}{\partial t} + \frac{k^2\alpha\theta\sin\frac{z}{\alpha}}{r}\frac{\partial}{\partial z} + r\cos\frac{z}{\alpha}\frac{\partial}{\partial\theta}, \qquad \mathbf{X_6} = k^2\theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial\theta},\\
&\mathbf{X_7} = -k^2\theta\sin\frac{z}{\alpha}\frac{\partial}{\partial r} - \frac{k^2\alpha\theta\cos\frac{z}{\alpha}}{r}\frac{\partial}{\partial z} + r\sin\frac{z}{\alpha}\frac{\partial}{\partial\theta}, \qquad \mathbf{X_8} = \sin\frac{z}{\alpha}\frac{\partial}{\partial r} + \frac{\alpha\cos\frac{z}{\alpha}}{r}\frac{\partial}{\partial z},\\
&\mathbf{X_9} = \cos\frac{z}{\alpha}\frac{\partial}{\partial r} - \frac{\alpha\sin\frac{z}{\alpha}}{r}\frac{\partial}{\partial z}, \quad \mathbf{Y_1} = 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + \theta\frac{\partial}{\partial\theta}, \quad \mathbf{Y_2} = s\frac{\partial}{\partial t}, \quad A_2 = 2t,\\
&\mathbf{Y_3} = s[s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + \theta\frac{\partial}{\partial\theta}], \quad A_3 = t^2 - r^2 - k^2\theta^2, \quad \mathbf{Y_4} = s[-\cos\frac{z}{\alpha}\frac{\partial}{\partial r} + \frac{\alpha\sin\frac{z}{\alpha}}{r}\frac{\partial}{\partial z}],\\
&A_4 = 2r\cos\frac{z}{\alpha}, \quad \mathbf{Y_5} = -s[\sin\frac{z}{\alpha}\frac{\partial}{\partial r} + \frac{\alpha\cos\frac{z}{\alpha}}{r}\frac{\partial}{\partial z}], \quad A_5 = 2r\sin\frac{z}{\alpha},\\
&\mathbf{Y_6} = s\frac{\partial}{\partial z}, \qquad A_6 = -2k^2\theta.
\end{aligned} \tag{5.9.6}$$

Table 5.23 contains the invariants of the above symmetries

Table 5.23: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2[-r\dot{t}\cos\frac{z}{\alpha} - \frac{\dot{z}tr\sin\frac{z}{\alpha}}{\alpha} + t\dot{r}\cos\frac{z}{\alpha}]$
$\mathbf{X_4}$	$\phi_5 = 2[-r\dot{t}\sin\frac{z}{\alpha} + \frac{\dot{z}tr\cos\frac{z}{\alpha}}{\alpha} + t\dot{r}\sin\frac{z}{\alpha}]$
$\mathbf{X_5}$	$\phi_6 = 2k^2[\theta\dot{t}\cos\frac{z}{\alpha} + \frac{\dot{z}r\theta\sin\frac{z}{\alpha}}{\alpha} + r\dot{\theta}\cos\frac{z}{\alpha}]$
$\mathbf{X_6}$	$\phi_7 = 2k^2[t\dot{\theta} - \theta\dot{t}]$
$\mathbf{X_7}$	$\phi_8 = 2k^2[r\dot{t}\sin\frac{z}{\alpha} - \frac{\dot{z}tr\cos\frac{z}{\alpha}}{\alpha} + t\dot{\theta}\sin\frac{z}{\alpha}]$
$\mathbf{X_8}$	$\phi_9 = 2[\dot{r}\sin\frac{z}{\alpha} + \dot{z}\frac{r\cos\frac{z}{\alpha}}{\alpha}]$
$\mathbf{X_9}$	$\phi_{10} = 2[\dot{r}\cos\frac{z}{\alpha} - \dot{z}\frac{r\sin\frac{z}{\alpha}}{\alpha}]$
$\mathbf{Y_1}$	$\phi_{11} = 2[s\mathcal{L} - t\dot{t} + r\dot{r} + k^2\theta\dot{\theta}]$
$\mathbf{Y_2}$	$\phi_{12} = 2[t - s\dot{t}]$
$\mathbf{Y_3}$	$\phi_{13} = s^2\mathcal{L} - 2s[t\dot{t} - r\dot{r} - k^2\theta\dot{\theta}] + t^2 - r^2 - k^2\theta^2$
$\mathbf{Y_4}$	$\phi_{14} = 2s[-\dot{r}\cos\frac{z}{\alpha} + \frac{\dot{z}r\sin\frac{z}{\alpha}}{\alpha}] + 2r\cos\frac{z}{\alpha}$
$\mathbf{Y_5}$	$\phi_{15} = -2s[\dot{r}\sin\frac{z}{\alpha} + \frac{\dot{z}r\cos\frac{z}{\alpha}}{\alpha}] + 2r\sin\frac{z}{\alpha}$
$\mathbf{Y_6}$	$\phi_{16} = 2k^2[s\dot{\theta} - \theta]$

Solution-IV:

All the coefficients of the metric are constant therefore it is the Minkowski spacetime. Components of Noether symmetry generators are

$$\eta^0 = c_5 r + c_8 z + c_{10}k^2\theta + c_2\frac{t}{2} + c_1 ts + c_{12}s + c_{16},$$
$$\eta^1 = c_4 + c_5 t - c_6 k^2\theta + c_9 z + c_2\frac{r}{2} + c_1 rs + c_{13}s,$$
$$\eta^2 = c_6 r + c_7 z + c_{10}t + c_2\frac{\theta}{2} + c_1\theta s + c_{11}s + c_{16},$$
$$\eta^3 = -c_7 k^2\theta + c_8 t - c_9 r + c_2\frac{z}{2} + c_1 zs + c_{14}s + c_{17},$$
$$A = c_1(t^2 - r^2 - k^2\theta^2 - z^2) - 2c_{11}\theta + 2c_{12}t - 2c_{13}r - 2c_{14}z + c_{18},$$
$$\xi = c_1 s^2 + c_2 s + c_3.$$

The metric in this case is

$$ds^2 = dt^2 - dr^2 - k^2 d\theta^2 - dz^2. \tag{5.9.7}$$

The additional Noether symmetry generators other than the minimal set are

$$\begin{aligned}
&\mathbf{X_3} = \frac{\partial}{\partial r}, \quad \mathbf{X_4} = r\frac{\partial}{\partial t} + t\frac{\partial}{\partial r}, \qquad \mathbf{X_5} = -k^2\theta\frac{\partial}{\partial r} + r\frac{\partial}{\partial \theta},\\
&\mathbf{X_6} = z\frac{\partial}{\partial \theta} - k^2\theta\frac{\partial}{\partial z}, \qquad \mathbf{X_7} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z},\\
&\mathbf{X_8} = z\frac{\partial}{\partial r} - r\frac{\partial}{\partial z}, \qquad \mathbf{X_9} = k^2\theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta},\\
&\mathbf{Y_1} = 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + \theta\frac{\partial}{\partial \theta} + z\frac{\partial}{\partial z}, \qquad \mathbf{Y_2} = s\frac{\partial}{\partial \theta}, A_2 = -2k^2\theta,\\
&\mathbf{Y_3} = s[s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + \theta\frac{\partial}{\partial \theta} + z\frac{\partial}{\partial z}], \qquad A_3 = t^2 - r^2 - k^2\theta^2 - z^2,\\
&\mathbf{Y_4} = s\frac{\partial}{\partial t}, \qquad A_4 = 2t, \quad \mathbf{Y_5} = s\frac{\partial}{\partial r}, \qquad A_5 = -2r, \qquad \mathbf{Y_6} = s\frac{\partial}{\partial z}, \qquad A_6 = -2z.
\end{aligned} \tag{5.9.8}$$

The first integrals corresponding to these Noether symmetry generators are given in Table 5.24.

Table 5.24: First Integrals

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\dot{r}$
$\mathbf{X_4}$	$\phi_5 = 2[t\dot{r} - r\dot{t}]$
$\mathbf{X_5}$	$\phi_6 = 2k^2[r\dot{\theta} - \theta\dot{r}]$
$\mathbf{X_6}$	$\phi_7 = 2k^2[z\dot{\theta} - \theta\dot{z}]$
$\mathbf{X_7}$	$\phi_8 = 2[t\dot{z} - z\dot{t}]$
$\mathbf{X_8}$	$\phi_9 = 2[z\dot{r} - r\dot{z}]$
$\mathbf{X_9}$	$\phi_{10} = 2k^2[t\dot{\theta} - \theta\dot{t}]$
$\mathbf{Y_1}$	$\phi_{11} = 2[s\mathcal{L} - t\dot{t} + r\dot{r} + k^2\theta\dot{\theta} + z\dot{z}]$
$\mathbf{Y_2}$	$\phi_{12} = 2k^2[s\dot{\theta} - \theta]$
$\mathbf{Y_3}$	$\phi_{13} = s^2\mathcal{L} - 2s[t\dot{t} - r\dot{r} - k^2\theta\dot{\theta} - z\dot{z}] + t^2 - r^2 - k^2\theta^2 - z^2$
$\mathbf{Y_4}$	$\phi_{14} = 2[t - s\dot{t}]$
$\mathbf{Y_5}$	$\phi_{15} = 2[s\dot{r} - r]$
$\mathbf{Y_6}$	$\phi_{16} = 2[s\dot{z} - z]$

Chapter 6

Noether Symmetries of the Arc Length Minimizing Lagrangian of Spherically Symmetric Static Spacetimes

6.1 Introduction

A spherical symmetric static spacetime has exactly three rotational Killing vector fields that preserve the metric forming $SO(3)$ as the isometry group. The study of spherically symmetric spacetimes is interesting as it helps in giving the understanding of phenomena of gravitational collapse and black holes, widely known subjects in the literature. For example the Schwarzschild solution is an exact solution of the Einstein field equations which is spherically symmetric that describes the gravitational field exterior to a static, spherical, uncharged point mass without angular momentum. The search for spherically symmetric spacetimes is an important task and due to their significance in understanding the dynamics around black holes, it is crucial to classify them with respect to their Noether symmetries and first integrals (conservation laws). Hence it would be interesting to find all these spacetimes along with a detailed characterization of the first integrals of the corresponding geodesic equations [49]. Besides the quantities which remain invariant under the geodesic motions yield important physical informations [23, 34, 44]. Classifications of plane symmetric, cylindrically symmetric and spherically symmetric spacetimes with respect to their Killing vectors, homotheties, Ricci collineations, curvature collineations have been done in references [13–15, 32, 35, 61, 62, 73].

The general form of a spherically symmetric static spacetime is [51]

$$ds^2 = e^{\nu(r)}dt^2 - e^{\mu(r)}dr^2 - e^{\lambda(r)}\mathrm{d}\Omega^2, \tag{6.1.1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$, and both ν and μ are arbitrary functions of radial coordinate 'r'. It is seen that $e^{\lambda(r)}$ can be one of the two forms (i) β^2 or (ii) r^2, where β is some constant [51] and can be absorbed in the definition of $\mathrm{d}\Omega^2$. We write down the determining equations using the corresponding Lagrangian density of the spacetime given in equation (6.1.1) and study the complete integrability of those equations for each case. The plan of the chapter is as follows. In Section 2 we discuss basic definitions and structure of Noether symmetries. In Section 3 we write down the determining equations for spherically symmetric static spacetimes which is a system of 19 linear PDEs. We obtain several cases for different values of 'ν' and 'μ' while integrating the PDEs that classify completely the spherically symmetric static spacetimes. We list different spacetimes according to different number of Noether symmetries in different sections. The characterization of first integrals along the geodesic motions is also carried out in each section.

6.2 Preliminaries

It is well-known that a general spherically symmetric static spacetime admits geodesic Lagrangian density [56]

$$\mathcal{L} = e^{\nu(r)}\dot{t}^2 - e^{\mu(r)}\dot{r}^2 - e^{\lambda(r)}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2), \tag{6.2.1}$$

where "$\dot{}$" denotes differentiation with respect to arc length parameter 's'.

For the Lagrangian given in equation (6.2.1) the Noether symmetry generator given in equation (3.5.8) takes the form

$$\mathbf{X} = \xi\frac{\partial}{\partial s} + \eta^0\frac{\partial}{\partial t} + \eta^1\frac{\partial}{\partial r} + \eta^2\frac{\partial}{\partial \theta} + \eta^3\frac{\partial}{\partial \phi}. \tag{6.2.2}$$

The Noether symmetry generator given in equation (6.2.2) leaves the action of a spherically symmetric static spacetime invariant. The coefficients of Noether symmetry namely ξ and η^i are functions of s, t, r, θ, ϕ. The coefficients of prolonged operator $\mathbf{X}^{[1]}$ that is η^i_s are functions of $s, t, r, \theta, \phi, \dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}$.

6.3 Determining PDEs and Computational Remarks

By substituting the value of Lagrangian (6.2.1), the corresponding symmetry generator and differential operator in equation (2.2.41) we obtained the following system of 19 PDEs

$$\begin{aligned}
&\xi_t = 0, \quad \xi_r = 0, \quad \xi_\theta = 0, \quad \xi_\phi = 0, \\
&A_s = 0, \quad A_t - 2e^{\nu(r)}\eta^0_s = 0, \quad A_r + 2\eta^1_s = 0, \\
&A_\theta + 2e^{\mu(x)}\eta^2_s = 0, \quad A_\phi + 2e^{\mu(r)}\eta^3_s = 0, \\
&\xi_s - \mu^{'}(r)\eta^1 - 2\eta^1_r = 0, \quad \xi_s - \nu^{'}(r)\eta^1 - 2\eta^0_t = 0 \\
&\xi_s - \frac{2}{r}\eta^1 - 2\eta^2_\theta = 0, \quad \xi_s - \frac{2}{r}\eta^1 - 2\cot\theta\eta^2 - 2\eta^3_\phi = 0, \\
&\eta^2_\phi + \sin^2\theta\eta^3_\theta = 0, \quad e^{\nu(r)}\eta^0_r - e^{\mu(x)}\eta^1_t = 0, \\
&e^{\mu(r)}\eta^1_\theta + r^2\eta^2_r = 0, \quad e^{\nu(r)}\eta^0_\theta - r^2\eta^2_t = 0, \\
&e^{\nu(r)}\eta^0_\phi - r^2\sin^2\theta\eta^3_t = 0, \quad e^{\mu(r)}\eta^1_\phi + r^2\sin^2\theta\eta^3_r = 0.
\end{aligned} \tag{6.3.1}$$

We intend to classify all Lagrangians of spherically symmetric static spacetimes with respect to their Noether symmetries by finding the solutions of the system of PDEs given by equations (6.3.1). In the following sections we enlist spherically symmetric static spacetimes, their Noether symmetries and relative first integrals. The Noether algebra of Noether symmetries are also presented here in the cases that are not known in the literature. In order to solve system of equations (6.3.1) of PDEs, it is noted that the first equation of system (6.3.1), simply implies that ξ can only be a function of arc length parameter s, i.e., $\xi(s)$. Distinct letter $\mathbf{Y}$ is used for those Noether symmetries which are not Killing vector fields. It is also remarked that a static spacetime always admits a time-like Killing vector field. Moreover, the Lagrangian given by equation (6.2.1) does not depend upon 't' explicitly therefore the time-like Killing vector field appears as a Noether symmetry in each case. Furthermore, the Lagrangian in equation (6.2.1) is spherically symmetric, therefore, the Lie algebra of Killing vector fields $so(3)$ corresponding to the Lie group $SO(3)$ is intrinsically admitted by each spacetime. Hence, we have the following five Noether symmetries as minimal set for spherically symmetric static spacetime

$$\mathbf{X_0} = \frac{\partial}{\partial t}, \quad \mathbf{Y_0} = \frac{\partial}{\partial s}, \tag{6.3.2}$$

$$\mathbf{X_1} = \frac{\partial}{\partial \phi}, \quad \mathbf{X_2} = \cos\phi \frac{\partial}{\partial \theta} - \cot\theta \sin\phi \frac{\partial}{\partial \phi}, \quad \mathbf{X_3} = \sin\phi \frac{\partial}{\partial \theta} + \cot\theta \cos\phi \frac{\partial}{\partial \phi}, \tag{6.3.3}$$

which form the basis of minimal 5−dimensional Noether algebra, in which $\mathbf{Y}_0$ is not a Killing vector field of spherically symmetric spacetime given in equation (6.1.1). The commutators relations of these five Noether symmetries are,

$[\mathbf{X_1}, \mathbf{X_2}] = -\mathbf{X_3}$, $[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}$, $[\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}$, $[\mathbf{X_i}, \mathbf{X_j}] = 0$ and $[\mathbf{X_i}, \mathbf{Y_0}] = 0$ otherwise, and is identified with the associated group $SO(3) \times \mathbb{R}^2$.

6.4 Five Noether Symmetries

Some examples of spacetimes whose action admit minimal set of Noether symmetries (five symmetries) appeared during the calculations for which $\lambda(r) = 2\ln r$ are given in Table 6.1.

Table 6.1: Metrics

No.	$\nu(r)$	$\mu(r)$
1.	$\ln\left(\frac{r}{\alpha}\right)^2$	arbitrary
2.	$\ln\left(1-(\frac{r}{\alpha})^2\right)$	arbitrary
3.	$\ln(\frac{r}{\alpha})^2$	$-\ln\left(1-(\frac{r}{\alpha})^2\right)$
4.	arbitrary	$-\ln(1-(\frac{r}{\alpha})^2)$
5.	$\ln(1-\frac{\alpha}{r})$	$-\ln(1-\frac{\alpha}{r})$

The Noether symmetries and corresponding first integrals are listed in the following Table 6.2.

Table 6.2: First Integrals

Gen	First Integrals
$\mathbf{X_0}$	$\phi_0 = -2e^{\nu(r)}\dot{t}$
$\mathbf{X_1}$	$\phi_1 = 2r^2 \sin^2\theta\dot{\phi}$
$\mathbf{X_2}$	$\phi_2 = 2r^2\left(\cos\phi\dot{\theta} - \cot\theta\sin\phi\dot{\phi}\right)$
$\mathbf{X_3}$	$\phi_3 = 2r^2\left(\sin\phi\dot{\theta} + \cot\theta\cos\phi\dot{\phi}\right)$
$\mathbf{Y_0}$	$\phi_4 = e^{\nu(r)}\dot{t}^2 - e^{\mu(r)}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) = \mathcal{L}$

with constant value of the gauge function, i.e., $A =$ constant.

6.5 Six Noether Symmetries

There are two distinct classes of spherically symmetric static spacetimes, the actions of the geodesic Lagrangians of which admit six Noether symmetries. The detail is given below in this section:

Solution-I:

Coefficients of the spacetime are

$$\nu(r) = k\ln\frac{r}{\alpha}, \quad \mu(r) = c.$$

Components of Noether symmetry generators are

$$\eta^0 = c_1\frac{2-k}{4}t + c_3, \quad \eta^1 = c_1\frac{r}{2}, \quad \eta^2 = c_4\cos\phi + c_5\sin\phi,$$
$$\eta^3 = -c_4\cot\theta\sin\phi + c_5\cot\theta\cos\phi + c_6, \quad \xi = c_1 s + c_2, \quad A = c_7.$$

The spacetime takes the form

$$\mathrm{d}s^2 = \left(\frac{r}{\alpha}\right)^k \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \quad \alpha \neq 0, \quad k \neq 0, 2 \tag{6.5.1}$$

which apart from minimal 5-dimensional Noether algebra also admit an additional Noether symmetry corresponding to the scaling transformation $(s, t, r) \longrightarrow (\lambda s, \lambda^p t, \lambda^{1/2} r)$, given by

$$\mathbf{Y_1} = s\frac{\partial}{\partial s} + pt\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r}, \quad p = \frac{2-k}{4} \tag{6.5.2}$$

forming a 6−dimensional Noether algebra. This induces a scale-invariant spherically symmetric static spacetime. The corresponding first integral is

$$\phi_6 = s\mathcal{L} - \frac{2-k}{2}\left(\frac{r}{a}\right)^k t\dot{t} + r\dot{r}. \tag{6.5.3}$$

Solution-II:

The metric coefficients are

$$\nu(r) = c, \quad \mu(r) \neq \ln\left(1 - \frac{r^2}{b^2}\right)^{-1}, \mu(r) \neq constant.$$

Components of Noether symmetry generators are

$$\eta^0 = c_2 s + c_3, \quad \eta^1 = 0, \quad \eta^2 = c_4 \cos\phi + c_5 \sin\phi,$$
$$\eta^3 = -c_4 \cot\theta \sin\phi + c_5 \cot\theta \cos\phi + c_6, \quad \xi = c_1, \quad A = 2c_2 t + c_7.$$

The spacetime in this case is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - e^{\mu(r)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \quad \mu(r) \neq \ln\left(1 - \frac{r^2}{b^2}\right)^{-1}, \mu(r) \neq constant, \quad b \neq 0. \tag{6.5.4}$$

The additional Noether symmetry and the relative non-trivial gauge term are

$$\mathbf{Y_1} = s\frac{\partial}{\partial t}, \quad A = 2t. \tag{6.5.5}$$

The first integral corresponding to $\mathbf{Y_1}$ is $\phi_6 = 2(t - s\dot{t})$.

6.6 Seven Noether Symmetries

There are five classes of spacetimes. Their actions of the geodesic Lagrangians admit seven Noether symmetries in which four classes admit the algebra of six Killing vector fields whereas one class admits only the minimal set of Killing vectors while the other two symmetries are Noether symmetries. We discuss them separately:

Solution-I:

Coefficients of the metric are

$$\nu(r) = \frac{r}{b}, \quad \mu(r) = c.$$

Components of Noether symmetry generators are

$$\eta^0 = c_2 - c_6(e^{\frac{-r}{b}} + \frac{t^2}{4b^2}) - c_7\frac{t}{2b}, \quad \eta^1 = c_6 t + c_7, \quad \eta^2 = c_3\cos\phi + c_4\sin\phi,$$
$$\eta^3 = -c_3\cot\theta\sin\phi + c_4\cot\theta\cos\phi + c_5, \quad \xi = c_1, \quad A = c_8.$$

The metric takes the form

$$\mathrm{d}s^2 = e^{r/b}\mathrm{d}t^2 - \mathrm{d}r^2 - \mathrm{d}\Omega^2, \quad b \neq 0.$$

The additional Noether symmetry generators are

$$\mathbf{X_4} = t\frac{\partial}{\partial r} - b\left(e^{-r/b} + \frac{t^2}{4b^2}\right)\frac{\partial}{\partial t} \quad , \quad \mathbf{X_5} = \frac{\partial}{\partial r} - \frac{t}{2b}\frac{\partial}{\partial t}.$$

The corresponding first integrals are

Table 6.3: First Integrals

Gen	First Integrals
$\mathbf{X_4}$	$\phi_5 = b\left(1 + \frac{t^2 e^{r/b}}{4b^2}\right)\dot{t} + t\dot{r}$
$\mathbf{X_5}$	$\phi_6 = \frac{t e^{r/b}}{b}\dot{t} + 2\dot{r}$

Solution-II:

Coefficients of the metric are

$$\nu(r) = 2\ln\sec\left(\frac{r}{a}\right) = \mu(r).$$

Components of symmetry generators are

$$\eta^0 = c_2 + c_6\sin(\frac{r}{a})\cos(\frac{t}{a}) + c_7\cos(\frac{r}{a})\cos(\frac{t}{a}), \quad \eta^1 = c_6\sin(\frac{t}{a})\cos(\frac{r}{a}) - c_7\sin(\frac{r}{a})\sin(\frac{t}{a}),$$
$$\eta^2 = c_3\cos\phi + c_4\sin\phi, \quad \eta^3 = -c_3\cot\theta\sin\phi + c_4\cot\theta\cos\phi + c_5, \quad \xi = c_1, \quad A = c_8.$$

The metric in this case is

$$\mathrm{d}s^2 = \sec^2\left(\frac{r}{a}\right)\mathrm{d}t^2 - \sec^2\left(\frac{r}{a}\right)\mathrm{d}r^2 - \mathrm{d}\Omega^2, \quad a \neq 0. \tag{6.6.1}$$

The symmetries other than the minimal set are

$$\mathbf{X_4} = \sin\left(\frac{r}{a}\right)\cos\left(\frac{t}{a}\right)\frac{\partial}{\partial t} + \sin\left(\frac{t}{a}\right)\cos\left(\frac{r}{a}\right)\frac{\partial}{\partial r},$$
$$\mathbf{X_5} = \cos\left(\frac{t}{a}\right)\cos\left(\frac{r}{a}\right)\frac{\partial}{\partial r} - \sin\left(\frac{r}{a}\right)\sin\left(\frac{t}{a}\right)\frac{\partial}{\partial t}.$$

The first integrals are

Table 6.4: First Integrals

Gen	First Integrals
$\mathbf{X_4}$	$\phi_5 = \sec^2\left(\frac{r}{a}\right)[-\dot{t}\sin\left(\frac{r}{a}\right)\cos\left(\frac{t}{a}\right) + \dot{r}\sin\left(\frac{t}{a}\right)\cos\left(\frac{r}{a}\right)]$
$\mathbf{X_5}$	$\phi_6 = \sec^2\left(\frac{r}{a}\right)[\dot{t}\sin\left(\frac{r}{a}\right)\sin\left(\frac{t}{a}\right) + \dot{r}\cos\left(\frac{t}{a}\right)\cos\left(\frac{r}{a}\right)\Big)]$

The corresponding Lie algebra is

$$[\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}, \quad [\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}, \quad , [\mathbf{X}_0, \mathbf{X_{4,2}}] = \frac{1}{a}\mathbf{X_{5,2}},$$
$$[\mathbf{X_0}, \mathbf{X_{5,2}}] = -\frac{1}{a}\mathbf{X_{4,2}}, \quad [\mathbf{X_i}, \mathbf{X_j}] = 0, \quad [\mathbf{X_i}, \mathbf{Y_0}] = 0, \quad \textit{otherwise}.$$

Solution-III:

Coefficients of the spacetime are

$$\nu(r) = \ln\left(1 - \frac{r^2}{b^2}\right), \quad \mu(r) = -\ln\left(1 - \frac{r^2}{b^2}\right).$$

Components of Noether symmetry generators are

$$\eta^0 = c_2 - c_6\frac{rbe^{\frac{t}{b}}}{\sqrt{r^2-b^2}} + c_7\frac{rbe^{\frac{-t}{b}}}{\sqrt{r^2-b^2}}, \quad \eta^1 = c_6 e^{\frac{t}{b}}\sqrt{r^2-b^2} + c_7 e^{\frac{-t}{b}}\sqrt{r^2-b^2},$$
$$\eta^2 = c_3\cos\phi + c_4\sin\phi, \quad \eta^3 = -c_3\cot\theta\sin\phi + c_4\cot\theta\cos\phi + c_5, \quad \xi = c_1, \quad A = c_8.$$

The spacetime here is

$$\mathrm{d}s^2 = \left(1 - \frac{r^2}{b^2}\right)\mathrm{d}t^2 - \left(1 - \frac{r^2}{b^2}\right)^{-1}\mathrm{d}r^2 - \mathrm{d}\Omega^2, \quad b \neq 0.$$

The additional Noether symmetry generators are

$$\mathbf{X_4} = -\frac{rbe^{\frac{t}{b}}}{\sqrt{r^2-b^2}}\frac{\partial}{\partial t} + \sqrt{r^2-b^2}e^{\frac{t}{b}}\frac{\partial}{\partial r}, \quad \mathbf{X_5} = \frac{rbe^{-t/b}}{\sqrt{r^2-b^2}}\frac{\partial}{\partial t} + \sqrt{r^2-b^2}e^{-t/b}\frac{\partial}{\partial r}.$$

The first integrals are

Table 6.5: First Integrals

Gen	First Integrals
$\mathbf{X_4}$	$\phi_5 = e^{\frac{t}{b}}[\frac{r\dot{t}\sqrt{r^2-b^2}}{b} + \frac{b^2\dot{r}}{\sqrt{r^2-b^2}}]$
$\mathbf{X_5}$	$\phi_6 = e^{\frac{-t}{b}}[-\frac{r\dot{t}\sqrt{r^2-b^2}}{b} + \frac{b^2\dot{r}}{\sqrt{r^2-b^2}}]$

Solution-IV:

Coefficients of the spacetime for seven Noether symmetries are

$$\nu(r) = -2\ln\left(\frac{r}{\alpha}\right) = \mu(r).$$

Components of the Noether symmetry generators are

$$\eta^0 = c_2 - c_6\frac{r^2+t^2}{2} + c_7 t, \quad \eta^1 = c_6 rt + c_7 r, \quad \eta^2 = c_3\cos\phi + c_4\sin\phi,$$

$$\eta^3 = -c_3\cot\theta\sin\phi + c_4\cot\theta\cos\phi + c_5, \quad \xi = c_1, \quad A = c_8.$$

The spacetime takes the form

$$\mathrm{d}s^2 = \left(\frac{\alpha}{r}\right)^2 \mathrm{d}t^2 - \left(\frac{\alpha}{r}\right)^2 \mathrm{d}r^2 - \mathrm{d}\Omega^2, \quad \alpha \neq 0. \tag{6.6.2}$$

The additional symmetries along with the minimal set of symmetries are

$$\mathbf{X_4} = \frac{t^2+r^2}{2}\frac{\partial}{\partial t} + rt\frac{\partial}{\partial r}, \quad \mathbf{X_5} = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}.$$

The first integral are given in the following table

Table 6.6: First Integrals

Gen	First Integrals
$\mathbf{X_4}$	$\phi_5 = -\frac{(t^2+r^2)\dot{t}}{r^2} + \frac{t\dot{r}}{r}$
$\mathbf{X_5}$	$\phi_6 = 2[-\frac{t\dot{t}}{r^2} + \dot{r}r]$

The Lie algebra in this case is

$$[\mathbf{X_1},\mathbf{X_2}] = -\mathbf{X_3}, \quad [\mathbf{X_1},\mathbf{X_3}] = \mathbf{X_2}, \quad [\mathbf{X_2},\mathbf{X_3}] = -\mathbf{X_1}, \quad ,[\mathbf{X_0},\mathbf{X_4}] = \mathbf{X_5},$$

$$[\mathbf{X_4},\mathbf{X_5}] = -\mathbf{X_4}, [\mathbf{X_0},\mathbf{X_5}] = \mathbf{X_0}, \quad [\mathbf{X_i},\mathbf{X_j}] = 0, \quad [\mathbf{X_i},\mathbf{Y_0}] = 0, \quad otherwise.$$

Solution-V:

Coefficients of the spacetime are

$$\nu(r) = 2\ln\left(\frac{r}{\alpha}\right), \quad \mu(r) = c.$$

Components of Noether symmetry generators are

$$\eta^0 = c_6, \quad \eta^1 = c_0 rs + c_1\frac{r}{2}, \quad \eta^2 = c_3\cos\phi + c_4\sin\phi, \quad A = -c_0 r^2 + c_8,$$

$$\eta^3 = -c_3\cot\theta\sin\phi + c_4\cot\theta\cos\phi + c_5, \quad \xi = c_0 s^2 + c_1 s + c_2.$$

The spacetime here is

$$\mathrm{d}s^2 = \left(\frac{r}{a}\right)^2 \mathrm{d}t^2 - \mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2, \quad a \neq 0. \tag{6.6.3}$$

Here we have the following two additional Noether symmetry generators

$$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{r}{2}\frac{\partial}{\partial r}, \quad \mathbf{Y_2} = \frac{s^2}{2}\frac{\partial}{\partial s} + \frac{rs}{2}\frac{\partial}{\partial r}, \quad A = \frac{-r^2}{2}.$$

First integrals corresponding $\mathbf{Y_1}$ and $\mathbf{Y_2}$ are given in Table 6.7.

Table 6.7: First Integrals

Gen	First Integrals
$\mathbf{Y_1}$	$\phi_5 = s\mathcal{L} + r\dot{r}$
$\mathbf{Y_2}$	$\phi_6 = \frac{1}{2}s^2\mathcal{L} + sr\dot{r} - \frac{1}{2}r^2$

6.7 Nine Noether Symmetries

This section contains some well known and important spacetimes. Here, we have four different classes of spacetimes in which the action of the geodesic Lagrangians admit nine Noether symmetries. Three classes contain two additional Noether symmetries and one class contains one extra Noether symmetry besides others which are all Killing vector fields. The detail of these spacetimes, their Noether symmetry generators and the corresponding first integral are given in this section:

Solution-I:

Coefficients of the spacetime are

$$\nu(r) = a, \quad \mu(r) = b.$$

Components of the Noether symmetry generators are

$$\eta^0 = c_2 + c_6 r + c_7 s, \quad \eta^1 = c_6 t + c_8 s + c_9,$$

$$\eta^2 = c_3 \cos\phi + c_4 \sin\phi, \quad A = 2c_6 t - 2c_8 r + c_{10},$$

$$\eta^3 = -c_3 \cot\theta \sin\phi + c_4 \cot\theta \cos\phi + c_5, \quad \xi = c_1.$$

The metric takes the form

$$ds^2 = dt^2 - dr^2 - \mathrm{d}\Omega^2.$$

The four additional Noether symmetry generators are

$$\mathbf{X_4} = r\frac{\partial}{\partial t} + t\frac{\partial}{\partial r}, \quad \mathbf{X_5} = \frac{\partial}{\partial r},$$
$$\mathbf{Y_1} = s\frac{\partial}{\partial t}, \quad A_1 = 2t \quad \mathbf{Y_2} = s\frac{\partial}{\partial r}, \quad A_2 = -2r.$$

The first integrals corresponding to these Noether symmetry generators are

Table 6.8: First Integrals

Gen	First Integrals
$\mathbf{X_4}$, $\mathbf{X_5}$	$\phi_5 = 2(t\dot{r} - r\dot{t})$, $\phi_6 = 2\dot{r}$
$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_7 = 2(t - s\dot{t})$, $\phi_8 = 2(s\dot{r} - r)$

Solution-II:

Coefficients of the spacetime are

$$\nu(r) = 2\ln\left(\frac{\beta}{r}\right), \quad \mu(r) = 4\ln\left(\frac{\beta}{r}\right).$$

Components of symmetry generators are

$$\eta^0 = c_2 - c_6\beta r e^{\frac{-t}{\beta}} + c_7\beta r e^{\frac{t}{\beta}} - c_8\frac{rse^{\frac{-t}{\beta}}}{\beta^3} + c_9\frac{rse^{\frac{t}{\beta}}}{\beta^3},$$
$$\eta^1 = c_6 r^2 e^{\frac{-t}{\beta}} + c_7 r^2 e^{\frac{t}{\beta}} + c_8\frac{r^2 se^{\frac{-t}{\beta}}}{\beta^4} + c_9\frac{r^2 se^{\frac{t}{\beta}}}{\beta^4},$$
$$\eta^2 = c_3\cos\phi + c_4\sin\phi, \quad A = c_8\frac{e^{\frac{-t}{\beta}}}{r} + c_9\frac{e^{\frac{t}{\beta}}}{r} + c_{10},$$
$$\eta^3 = -c_3\cot\theta\sin\phi + c_4\cot\theta\cos\phi + c_5, \quad \xi = c_1.$$

The metric takes the form

$$ds^2 = \left(\frac{\beta}{r}\right)^2 dt^2 - \left(\frac{\beta}{r}\right)^4 dr^2 - \mathrm{d}\Omega^2, \quad \beta \neq 0. \tag{6.7.1}$$

The four additional Noether symmetry generators are

$$\mathbf{X_4} = -\beta r e^{\frac{-t}{\beta}}\frac{\partial}{\partial t} + r^2 e^{\frac{-t}{\beta}}\frac{\partial}{\partial r}, \qquad \mathbf{X_5} = \beta r e^{\frac{t}{\beta}}\frac{\partial}{\partial t} + r^2 e^{\frac{t}{\beta}}\frac{\partial}{\partial r}$$

$$\mathbf{Y_1} = -\frac{rse^{\frac{-t}{\beta}}}{\beta^3}\frac{\partial}{\partial t} + \frac{r^2 se^{\frac{-t}{\beta}}}{\beta^4}\frac{\partial}{\partial r}, \quad A_1 = \frac{2e^{\frac{-t}{\beta}}}{r}$$

$$\mathbf{Y_2} = \frac{rse^{\frac{t}{\beta}}}{\beta^3}\frac{\partial}{\partial t} + \frac{r^2 se^{\frac{t}{\beta}}}{\beta^4}\frac{\partial}{\partial r}, \quad A_2 = \frac{2e^{\frac{t}{\beta}}}{r}.$$

The first integrals are

Table 6.9: First Integrals

Gen	First Integrals
$\mathbf{X_4}, \quad \mathbf{X_5}$	$\phi_5 = 2e^{\frac{-t}{\beta}}\beta^3[\frac{\dot{t}}{r} + \frac{\beta\dot{r}}{r^2}], \qquad \phi_6 = 2e^{\frac{t}{\beta}}\beta^3[-\frac{\dot{t}}{r} + \frac{\beta\dot{r}}{r^2}]$
$\mathbf{Y_1}, \quad \mathbf{Y_2}$	$\phi_7 = 2se^{\frac{-t}{\beta}}[\frac{\dot{t}}{r\beta} + \frac{\dot{r}}{r^2}] + \frac{2e^{\frac{-t}{\beta}}}{r}, \quad \phi_8 = 2se^{\frac{t}{\beta}}[-\frac{\dot{t}}{\beta r} + \frac{\dot{r}}{r^2}] + \frac{2e^{\frac{t}{\beta}}}{r}$

The Lie algebra is

$$[\mathbf{X_1}, \mathbf{X_2}] = -\mathbf{X_3}, \quad [\mathbf{X_1}, \mathbf{X_3}] = \mathbf{X_2}, \quad [\mathbf{X_2}, \mathbf{X_3}] = -\mathbf{X_1}, \quad ,[\mathbf{X_0}, \mathbf{X_4}] = \frac{1}{\alpha}\mathbf{X_4},$$
$$[\mathbf{X_4}, \mathbf{X_5}] = -\mathbf{X_4} \quad [\mathbf{X_0}, \mathbf{X_5}] = \frac{1}{\alpha}\mathbf{X_5}, \quad [\mathbf{X_0}, \mathbf{Y_1}] = -\frac{1}{\alpha}\mathbf{Y_1}, \quad [\mathbf{X_0}, \mathbf{Y_2}] = \frac{1}{\alpha}\mathbf{Y_2},$$
$$[\mathbf{Y_0}, \mathbf{Y_1}] = \frac{1}{\alpha^4}\mathbf{X_4}, \quad [\mathbf{Y_0}, \mathbf{Y_2}] = \frac{1}{\alpha^4}\mathbf{X_5}, \quad [\mathbf{X_i}, \mathbf{X_j}] = 0, \quad [\mathbf{X_i}, \mathbf{Y_0}] = 0,$$
$$[\mathbf{Y_i}, \mathbf{Y_j}] = 0, \qquad \textit{otherwise}.$$

Solution-III:

Coefficients of the metric are

$$\nu(r) = 2\ln\left(1 + \frac{r}{b}\right), \quad \mu(r) = c.$$

Components of the Noether symmetry generators are

$$\eta^0 = c_2 - c_6\frac{be^{\frac{-t}{b}}}{b+r} - c_7\frac{be^{\frac{t}{b}}}{b+r} - c_8\frac{bse^{\frac{-t}{b}}}{2(b+r)} + c_9\frac{bse^{\frac{t}{b}}}{2(b+r)},$$
$$\eta^1 = c_6 e^{\frac{-t}{b}} + c_7 e^{\frac{t}{b}} - c_8\frac{se^{\frac{-t}{b}}}{2} - c_9\frac{se^{\frac{t}{b}}}{2},$$
$$\eta^2 = c_3\cos\phi + c_4\sin\phi, \quad A = c_8 e^{\frac{-t}{b}}(b+r) + c_9 e^{\frac{t}{\beta}}(b+r) + c_{10},$$
$$\eta^3 = -c_3\cot\theta\sin\phi + c_4\cot\theta\cos\phi + c_5, \quad \xi = c_1.$$

The spacetime takes the form

$$ds^2 = \left(1+\frac{r}{b}\right)^2 dt^2 - dr^2 - \mathrm{d}\Omega^2, \quad b \neq 0.$$

The Noether symmetry generators other than the minimal set are

$$\begin{aligned}
\mathbf{X_4} &= \frac{b}{b+r}e^{\frac{-t}{b}}\frac{\partial}{\partial t} + e^{\frac{-t}{b}}\frac{\partial}{\partial r}, \quad \mathbf{X_5} = -\frac{b}{b+r}e^{\frac{t}{b}}\frac{\partial}{\partial t} + e^{\frac{t}{b}}\frac{\partial}{\partial r},\\
\mathbf{Y_1} &= -\frac{bs}{2(b+r)}e^{\frac{-t}{b}}\frac{\partial}{\partial t} - \frac{s}{2}e^{\frac{-t}{b}}\frac{\partial}{\partial r}, \quad A_1 = (b+r)e^{\frac{-t}{b}}\\
\mathbf{Y_2} &= \frac{bs}{2(b+r)}e^{\frac{t}{b}}\frac{\partial}{\partial t} - \frac{s}{2}e^{\frac{t}{b}}\frac{\partial}{\partial r}, \quad A_2 = (b+r)e^{\frac{t}{b}}.
\end{aligned}$$

The corresponding first integrals are

Table 6.10: First Integrals

Gen	First Integrals
$\mathbf{X_4}$, $\mathbf{X_5}$	$\phi_5 = e^{\frac{-t}{b}}\left(-\frac{\dot{t}(b+r)}{b} + \dot{r}\right)$, $\phi_6 = e^{\frac{t}{b}}\left(\frac{\dot{t}(b+r)}{b} + \dot{r}\right)$
$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_7 = e^{\frac{-t}{b}}\{\frac{s\dot{t}(b+r)}{b} - s\dot{r} + (b+r)\}$, $\phi_8 = e^{\frac{t}{b}}\{-\frac{s\dot{t}(b+r)}{b} - s\dot{r} + (b+r)\}$

Solution-IV:

Coefficients of the spacetime are

$$\nu(r) = c, \quad \mu(r) = -\ln\left(1-\frac{r^2}{b^2}\right).$$

The values of functions η^i, $i = 0,1,2,3$, ξ and A are

$$\begin{aligned}
&\eta^0 = c_2 + c_9 s, \quad \eta^1 = c_6\sqrt{b^2-r^2}\cos\theta\sin\phi + c_7\sqrt{b^2-r^2}\sin\theta\cos\phi + c_8\sqrt{b^2-r^2}\cos\theta,\\
&\eta^2 = c_3\cos\phi + c_4\sin\phi - c_6\frac{\sqrt{b^2-r^2}}{r}\cos\theta\sin\phi - c_7\frac{\sqrt{b^2-r^2}}{r}\cos\theta\cos\phi - c_8\frac{\sqrt{b^2-r^2}}{r}\sin\theta,\\
&\eta^3 = -c_3\cot\theta\sin\phi + c_4\cot\theta\cos\phi + c_6\frac{\sqrt{b^2-r^2}}{r\sin\theta}\cos\phi - c_7\frac{\sqrt{b^2-r^2}}{r\sin\theta}\sin\phi + c_5,\\
&\xi = c_1, \quad A = c_9 t + c_{10}.
\end{aligned}$$

The corresponding spacetime is

$$ds^2 = dt^2 - \frac{dr^2}{1-\frac{r^2}{b^2}} - r^2\mathrm{d}\Omega^2, \quad b \neq 0.$$

Which is the Einstein universe. The Noether symmetry generators other than the minimal set are

$$\mathbf{X_4} = \sqrt{b^2-r^2}\sin\phi\sin\theta\frac{\partial}{\partial r} - \frac{\sqrt{b^2-r^2}}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{\sqrt{b^2-r^2}}{r\sin\theta}\cos\phi\frac{\partial}{\partial\phi},$$
$$\mathbf{X_5} = \sqrt{b^2-r^2}\cos\phi\sin\theta\frac{\partial}{\partial r} - \frac{\sqrt{b^2-r^2}}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sqrt{b^2-r^2}}{r\sin\theta}\sin\phi\frac{\partial}{\partial\phi},$$
$$\mathbf{X_6} = \sqrt{b^2-r^2}\cos\theta\frac{\partial}{\partial r} - \frac{\sqrt{b^2-r^2}}{r}\sin\theta\frac{\partial}{\partial\theta}, \quad \mathbf{Y_1} = s\frac{\partial}{\partial t}, \qquad A = 2t.$$

The first integrals are

Table 6.11: First Integrals

Gen	First Integrals
$\mathbf{X_4}$	$\phi_6 = \frac{b^2\dot{r}\sin\phi\sin\theta}{\sqrt{b^2-r^2}} - r\dot{\theta}\sqrt{b^2-r^2}\cos\theta\sin\phi + r\dot{\phi}\sqrt{b^2-r^2}\sin\theta\cos\phi$
$\mathbf{X_5}$	$\phi_6 = \frac{b^2\dot{r}\cos\phi\sin\theta}{\sqrt{b^2-r^2}} - r\dot{\theta}\sqrt{b^2-r^2}\cos\theta\cos\phi - r\dot{\phi}\sqrt{b^2-r^2}\sin\theta\sin\phi$
$\mathbf{X_6}$,$\mathbf{Y_1}$	$\phi_7 = \frac{b^2\dot{r}\cos\theta}{\sqrt{b^2-r^2}} - r\dot{\theta}\sqrt{b^2-r^2}\sin\theta, \quad \phi_8 = 2(t-s\dot{t})$

6.8 Eleven Noether Symmetries

The famous de-Sitter metric turns out to be the only one, the action of the Lagrangian of which admit eleven Noether symmetries. Except $\mathbf{Y_0}$ all others are the Killing vectors:

Solution:

the metric coefficients are

$$\nu(r) = \ln\left(1-\frac{r^2}{b^2}\right), \quad \mu(r) = -\ln\left(1-\frac{r^2}{b^2}\right).$$

Coefficients of Noether symmetry generators are

$$\eta^0 = c_2 + c_6\frac{br\sin\phi\sin\theta\cos(t/b)}{\sqrt{b^2-r^2}} + c_7\frac{br\cos\phi\sin\theta\cos(t/b)}{\sqrt{b^2-r^2}} - c_8\frac{br\sin\phi\sin\theta\sin(t/b)}{\sqrt{b^2-r^2}}$$
$$- c_9\frac{br\cos\phi\sin\theta\sin(t/b)}{\sqrt{b^2-r^2}} + c_{10}\frac{br\cos\theta\cos(t/b)}{\sqrt{b^2-r^2}} - c_{11}\frac{br\cos\theta\sin(t/b)}{\sqrt{b^2-r^2}},$$
$$\eta^1 = c_6\sqrt{b^2-r^2}\sin\theta\sin\phi\sin(t/b) + c_7\sqrt{b^2-r^2}\sin\theta\cos\phi\sin(t/b) + c_8\sqrt{b^2-r^2}\sin\theta\sin\phi\cos(t/b)$$
$$+ c_9\sqrt{b^2-r^2}\sin\theta\cos\phi\cos(t/b) + c_{10}\sqrt{b^2-r^2}\cos\theta\sin(t/b) + +c_{11}\sqrt{b^2-r^2}\cos\theta\cos(t/b),$$

$$\eta^2 = c_3\cos\phi + c_4\sin\phi + c_6 r\sqrt{b^2-r^2}\cos\theta\sin\phi\sin(t/b) - c_7\frac{\sqrt{b^2-r^2}}{r}\cos\theta\cos\phi\sin(t/b)$$
$$- c_8 r\sqrt{b^2-r^2}\cos\theta\sin\phi\cos(t/b) + c_9\frac{\sqrt{b^2-r^2}}{r}\cos\theta\cos\phi\cos(t/b)$$
$$- c_{10}\frac{\sqrt{b^2-r^2}}{r}\sin\theta\sin(t/b) - c_{11}\frac{\sqrt{b^2-r^2}}{r}\sin\theta\cos(t/b),$$
$$\eta^3 = -c_3\cot\theta\sin\phi + c_4\cot\theta\cos\phi + c_6\frac{\sqrt{b^2-r^2}}{r\sin\theta}\cos\phi\sin(t/b) - c_7\frac{\sqrt{b^2-r^2}}{r\sin\theta}\sin\phi\sin(t/b)$$
$$+ c_8\frac{\sqrt{b^2-r^2}}{r\sin\theta}\cos\phi\cos(t/b) - c_9\frac{\sqrt{b^2-r^2}}{r\sin\theta}\sin\phi\cos(t/b) + c_5, \quad \xi = c_1, \quad A = c_{12}.$$

The corresponding spacetime is

$$\mathrm{d}s^2 = \left(1-\frac{r^2}{b^2}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{\left(1-\frac{r^2}{b^2}\right)} - r^2\mathrm{d}\Omega^2, \quad b \neq 0. \tag{6.8.1}$$

We have the following Noether symmetry generators along with the minimal set of Noether symmetries for the metric given by equations (6.8.1)

$$\mathbf{X_4} = \frac{br\sin\phi\sin\theta\cos(t/b)}{\sqrt{b^2-r^2}}\frac{\partial}{\partial t} + \sin(t/b)\sqrt{b^2-r^2}\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + r\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right),$$
$$\mathbf{X_5} = \frac{br\cos\phi\sin\theta\cos(t/b)}{\sqrt{b^2-r^2}}\frac{\partial}{\partial t} + \sin(t/b)\sqrt{b^2-r^2}\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right),$$
$$\mathbf{X_6} = \frac{-br\sin\phi\sin\theta\sin(t/b)}{\sqrt{b^2-r^2}}\frac{\partial}{\partial t} + \cos(t/b)\sqrt{b^2-r^2}\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + r\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right),$$
$$\mathbf{X_7} = \frac{-br\cos\phi\sin\theta\sin(t/b)}{\sqrt{b^2-r^2}}\frac{\partial}{\partial t} + \cos(t/b)\sqrt{b^2-r^2}\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right),$$
$$\mathbf{X_8} = \frac{br\cos\theta\cos(t/b)}{\sqrt{b^2-r^2}}\frac{\partial}{\partial t} + \sin(t/b)\sqrt{b^2-r^2}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right),$$
$$\mathbf{X_9} = \frac{-br\cos\theta\sin(t/b)}{\sqrt{b^2-r^2}}\frac{\partial}{\partial t} + \cos(t/b)\sqrt{b^2-r^2}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right).$$

The first integrals corresponding to these Noether symmetries are given in Table 6.12.

6.9 Seventeen Noether Symmetries

For seventeen Noether symmetry only one spherically symmetric static spacetime is obtain which is the famous Minkowski metric

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

that represents a flat spacetime and admits seventeen Noether symmetries. The list of all Noether symmetries and the corresponding first integrals (cartesian coordinates) are given in [1, 40].

Table 6.12: First Integrals

Gen	First Integrals
$\mathbf{X_4}$	$\phi_5 = -\frac{r}{b}\sqrt{b^2-r^2}\sin\phi\sin\theta\cos(t/b)\dot{t} + \frac{b^2}{\sqrt{b^2-r^2}}\sin\phi\sin\theta\sin(t/b)\dot{r} + r\sqrt{b^2-r^2}\cos\theta\sin\phi\sin(t/b)\dot{\theta} + r\sqrt{b^2-r^2}\sin\theta\cos\phi\sin(t/b)\dot{\phi}$
$\mathbf{X_5}$	$\phi_6 = -\frac{r}{b}\sqrt{b^2-r^2}\cos\phi\sin\theta\cos(t/b)\dot{t} + \frac{b^2}{\sqrt{b^2-r^2}}\cos\phi\sin\theta\sin(t/b)\dot{r} + r\sqrt{b^2-r^2}\cos\theta\cos\phi\sin(t/b)\dot{\theta} - r\sqrt{b^2-r^2}\sin\theta\sin\phi\sin(t/b)\dot{\phi}$
$\mathbf{X_6}$	$\phi_7 = \frac{r}{b}\sqrt{b^2-r^2}\sin\phi\sin\theta\sin(t/b)\dot{t} + \frac{b^2}{\sqrt{b^2-r^2}}\sin\phi\sin\theta\cos(t/b)\dot{r} + r\sqrt{b^2-r^2}\cos\theta\sin\phi\cos(t/b)\dot{\theta} + r\sqrt{b^2-r^2}\sin\theta\cos\phi\cos(t/b)\dot{\phi}$
$\mathbf{X_7}$	$\phi_8 = \frac{r}{b}\sqrt{b^2-r^2}\cos\phi\sin\theta\cos(t/b)\dot{t} + \frac{b^2}{\sqrt{b^2-r^2}}\cos\phi\sin\theta\sin(t/b)\dot{r} + r\sqrt{b^2-r^2}\cos\theta\cos\phi\sin(t/b)\dot{\theta} - r\sqrt{b^2-r^2}\sin\theta\sin\phi\sin(t/b)\dot{\phi}$
$\mathbf{X_8}$	$\phi_9 = -\frac{r}{b}\sqrt{b^2-r^2}\cos\theta\cos(t/b)\dot{t} + \frac{b^2}{\sqrt{b^2-r^2}}\cos\theta\sin(t/b)\dot{r} - r\sqrt{b^2-r^2}\sin\theta\sin(t/b)\dot{\theta}$
$\mathbf{X_9}$	$\phi_{10} = \frac{r}{b}\sqrt{b^2-r^2}\cos\theta\sin(t/b)\dot{t} + \frac{b^2}{\sqrt{b^2-r^2}}\cos\theta\cos(t/b)\dot{r} - r\sqrt{b^2-r^2}\sin\theta\cos(t/b)\dot{\theta}$

Chapter 7

Conclusion

In this thesis Noether symmetries of the arc length minimizing Lagrangian densities of plane symmetric, cylindrically symmetric and spherically symmetric static spacetimes were obtained. For this purpose the general arc length minimizing Lagrangians of the general plane symmetric, cylindrically symmetric and spherically symmetric static spacetimes were used in the Noether symmetry equation to obtain systems of 19 PDEs in each case. Solutions of these systems provided a classification of the arc length minimizing Lagrangians of plane symmetric, cylindrically symmetric and spherically symmetric static spacetimes. This classification provided us with exact solutions of Einstein's field equations, the Noether symmetries and the corresponding first integrals. The first integrals were further used to investigate the conservation laws in each spacetime.

In Chapter 1, a brief introduction to Lie point symmetries of differential equations was given. Concepts of the contact symmetry transformations, Lie Backlund symmetry transformations and approximate Lie symmetry transformations were also given in the same chapter. Introduction to variational problems, Noether symmetry transformations, Noether symmetry equation and Euler-Lagrange equations were given in Chapter 2.

In Chapter 3, the classification of plane symmetric static spacetimes by Noether symmetries was presented. The approximate Noether symmetries and the corresponding approximate conservation laws of time conformal plane symmetric spacetimes were presented in Chapter 4. The Noether symmetries and their first integrals of cylindrically symmetric static spacetime were given in Chapter 5 while the Noether symmetries and the corresponding first integrals of spherically symmetric static spacetime were given in Chapter 6.

7.1 Plane Symmetric Spacetimes and Noether Symmetries

For the classification purpose the arc length minimizing Lagrangian density of the general plane symmetric static spacetime is used in the Noether symmetry equation to get system of 19 PDEs. Solutions of this system give us complete classification of plane symmetric static spacetimes by Noether symmetries. It turns out that for the action of arc length minimizing Lagrangian densities of plane symmetric static spacetimes, there exist 5, 6, 7, 8, 9, 11 or 17 Noether symmetries. Some cases having minimum number of the Noether symmetries (5 Noether symmetries) are given in Table 3.1. These metrics admit minimal set of isometries (i.e. 4 isometries).

Metrics for which the corresponding actions admit 6 Noether symmetries are given by equations (3.5.1), (3.5.2), (3.5.3), (3.5.5), (3.5.6) and (3.5.7). Metric given by equation (3.5.1) has 5 isometries while others admit the minimal set of isometries. The metrics given by equations (3.6.1), (3.6.5) and (3.6.7) admit different sets of 7 Noether symmetries. The metric of equation (3.6.1) admits 6 isometries while equations (3.6.5) and (3.6.7) admit only the minimal set of isometries. The spacetimes for which the actions admit 8 Noether symmetries are given by equations (3.7.1), (3.7.4) and (3.7.5). The action of the arc length minimizing Lagrangian densities of the Einstein spacetime, the Bertotti-Robinson like spacetimes and the metric given by equation (3.8.11) admit 9 Noether symmetries. These metrics respectively have 7, 6, and 4 isometries. The De-Sitter spacetime admits 11 Noether symmetries (10 isometries and $\mathbf{Y_0}$) and the maximum number (i.e. 17) of Noether symmetries appears for the Minkowski spacetime. Tables (3.2)-(3.15) provide conserved forms or first integrals corresponding to the Noether symmetries. Three cases of plane symmetric static spacetime are obtained which are new in the literature. These spacetimes are given in equations (3.5.5), (3.6.5) and (3.7.4). The list of their surviving components of the Riemann curvature tensors, the Ricci curvature tensors and the Ricci scalars are given below:

$$(3.5.5): \quad R_{0202} = -\frac{(\frac{x}{\alpha})^b b}{2\alpha^2} = R_{0303}, \quad R_{1212} = \frac{(\frac{x}{\alpha})^b b(b-2)}{4x^2} = R_{1313}, \quad R_{2323} = \frac{(\frac{x}{\alpha})^{2b} b^2}{4x^2};$$

$$R_{00} = -\frac{b}{\alpha^2}, \quad R_{11} = \frac{b(b-2)}{2x^2}, \quad R_{22} = \frac{(\frac{x}{\alpha})^b b^2}{2x^2} = R_{33}; \quad R_s = -\frac{3b^2}{2x^2},$$

$$(3.6.5): \quad R_{1212} = \frac{(\frac{x}{\alpha})^b b(b-2)}{4x^2} = R_{1313}, \quad R_{2323} = \frac{(\frac{x}{\alpha})^{2b} b^2}{4x^2};$$

$$R_{11} = \frac{b(b-2)}{2x^2}, \quad R_{22} = \frac{(\frac{x}{\alpha})^2 b(b-1)}{2x^2}; \quad R_s = -\frac{b(3b-4)}{2x^2},$$

$$(3.7.4): \quad R_{2323} = \frac{x^2}{\alpha^4}; \quad R_{22} = \frac{1}{\alpha^2} = R_{33}; \quad R_s = -\frac{2}{x^2}.$$

7.2 Time Conformal Plane Symmetric Spacetime and Noether Symmetries

In Chapter 4, approximate Noether symmetries of the action of time conformal plane symmetric spacetimes are given. Three types of approximate Noether symmetries are obtained. The first one is time-like Killing vector field which corresponds to the energy content of the given spacetimes. The second symmetry which carries approximate part is the scaling symmetry, and the third one corresponds to the Lorentz transformations. Using these approximate symmetries we also find the approximate first integrals which correspond to the approximate conservation laws in the respective spacetimes. Although the spacetimes given in this chapter are not the exact gravitational wave spacetimes but can be considered approximate gravitational wave spacetimes. In this chapter the spacetimes which admit the approximation along with approximate Noether symmetries and the corresponding first integrals are presented.

7.2.1 Plane symmetric Static Vacuum Solutions of EFEs

Consider the following general plane symmetric static spacetime

$$ds^2 = e^{\nu(x)}dt^2 - dx^2 - e^{\mu(x)}(dy^2 + dz^2). \tag{7.2.1}$$

Equating the surviving components of the Ricci tensor for this spacetime to zero, we get a system of three non-linear PDEs in two unknown functions $\nu(x)$ and $\mu(x)$,

$$\begin{aligned}
R_{00} &= 2\nu^{''}(x) + 2\nu^{'}(x)\mu^{'}(x) + \nu^{'2}(x) = 0,\\
R_{11} &= 2\nu^{''}(x) + 4\mu^{''}(x) + \nu^{'2}(x) + 2\mu^{'2}(x) = 0,\\
R_{22} &= 2\mu^{''}(x) + 2\mu^{'2}(x) + \nu^{'}(x)\mu^{'}(x) = 0.
\end{aligned} \tag{7.2.2}$$

The solution of this system is

$$\nu(x) = -\frac{2}{3}\ln\left(\frac{x}{\alpha}\right), \quad \mu(x) = \frac{4}{3}\ln\left(\frac{x}{\alpha}\right), \tag{7.2.3}$$

which discribs the famous Taub spacetime having singularity at $x = 0$ [67]. This is a static vacuum solution of EFEs and does not admit time conformal perturbation and hence is not actual non-static gravitational wave spacetime [44]. The coefficients of the metric defined in equation (7.2.3) is a special case of the metric given in equation (3.5.7).

7.3 Cylindrically Symmetric Spacetimes and Noether Symmetries

The Noether symmetries of the action of cylindrically symmetric static spacetimes are given in Chapter 5. To get all possible metrics, geodesic Lagrangian density for the general cylindrically symmetric static metric has been considered. It has been observed that there may be 5, 6, 7, 8, 9, 11, and 17 Noether symmetries for the geodesic Lagrangian of cylindrically symmetric static spacetimes. There are infinite number of metrics for which the actions of the corresponding Lagrangians admit four or five Noether symmetries. We have twenty four classes for six Noether symmetries, three classes for seven Noether symmetries, seven for eight Noether symmetries, four for nine Noether symmetries and one for eleven Noether symmetries. There are 4 classes of 17 Noether symmetries in which one spacetime is the Minkowaski spacetime. The first integrals in each case are also given correspondingly in tabulated form. It is important to note that in this classification, all the metrics given in [60, 62] have been recovered.

7.3.1 Some Cases of Cylindrically Symmetric Vacuum Solutions

We discuss two cylindrically symmetric static vacuum solutions of EFEs [2]:

(i): The metric

$$ds^2 = \left(\frac{r}{\alpha}\right)^{\frac{1-\sqrt{5}}{2}} dt^2 - dr^2 - \left(\frac{r}{\alpha}\right) d\theta^2 - \left(\frac{r}{\alpha}\right)^{\frac{1+\sqrt{5}}{2}} dz^2, \quad \alpha \neq 0, \tag{7.3.1}$$

represent the vacuum solution. The corresponding geodesic Lagrangian density takes the form

$$\mathcal{L} = \left(\frac{r}{\alpha}\right)^{\frac{1-\sqrt{5}}{2}} \dot{t}^2 - \dot{r}^2 - \left(\frac{r}{\alpha}\right) \dot{\theta}^2 - \left(\frac{r}{\alpha}\right)^{\frac{1+\sqrt{5}}{2}} \dot{z}^2. \tag{7.3.2}$$

The action of Lagrangian given in equation (7.3.2) admits five Noether symmetry generators

$$\begin{aligned} &\mathbf{Y}_0 = \frac{\partial}{\partial s}, \quad \mathbf{X}_0 = \frac{\partial}{\partial t}, \quad \mathbf{X}_1 = \frac{\partial}{\partial \theta}, \quad \mathbf{X}_2 = \frac{\partial}{\partial z}, \\ &\mathbf{X}_3 = s\frac{\partial}{\partial s} + \frac{t(3+\sqrt{5})}{8}\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{\theta}{4}\frac{\partial}{\partial \theta} + \frac{z(3-\sqrt{5})}{8}\frac{\partial}{\partial z}. \end{aligned} \tag{7.3.3}$$

Using these Noether symmetries in the expression given in equation (3.2.5), following table of conservation laws is obtained [2, 22].

Table 7.1: First Integrals

Gen	First Integrals
$\mathbf{X}_0$	$\phi_1 = \mathcal{L}$
$\mathbf{X}_0$	$\phi_1 = -2\dot{t}(\frac{r}{\alpha})^{\frac{1-\sqrt{5}}{2}}$
$\mathbf{X}_1$	$\phi_1 = 2b^2\dot{\theta}\frac{r}{\alpha}$
$\mathbf{X}_2$	$\phi_1 = 2\dot{z}(\frac{r}{\alpha})^{\frac{1+\sqrt{5}}{2}}$
$\mathbf{X}_3$	$\phi_1 = s\mathcal{L} - \frac{t\dot{t}(3+\sqrt{5})}{4}(\frac{r}{\alpha})^{\frac{1+\sqrt{5}}{2}} + r\dot{r} + \frac{b^2 r\theta\dot{\theta}}{2\alpha} + \frac{z\dot{z}(3-\sqrt{5})}{4}(\frac{r}{\alpha})^{\frac{1-\sqrt{5}}{2}}$

The non-zero components of the Riemann curvature tensor for the spacetime given in equation (7.3.1) are

$$\begin{aligned} &R_{0101} = -\frac{(\frac{r}{\alpha})^{\frac{1-\sqrt{5}}{2}}(1+\sqrt{5})}{8r^2}, \quad R_{0202} = \frac{(\frac{r}{\alpha})^{\frac{1-\sqrt{5}}{2}}(-1+\sqrt{5})}{8\alpha r}, \quad R_{0303} = \frac{1}{4\alpha r}, \\ &R_{1212} = -\frac{1}{4\alpha r}, \quad R_{1313} = \frac{(\frac{r}{\alpha})^{\frac{1-\sqrt{5}}{2}}(1+\sqrt{5})}{8r^2}, \quad R_{2323} = \frac{(\frac{r}{\alpha})^{\frac{1+\sqrt{5}}{2}}(1+\sqrt{5})}{8\alpha r}. \end{aligned}$$

Another cylindrically symmetric static vacuum solution is

$$(ii): \quad ds^2 = \left(\frac{r}{\alpha}\right)^{\frac{1+\sqrt{5}}{2}} dt^2 - dr^2 - \left(\frac{r}{\alpha}\right) d\theta^2 - \left(\frac{r}{\alpha}\right)^{\frac{1-\sqrt{5}}{2}} dz^2, \quad \alpha \neq 0. \tag{7.3.4}$$

The corresponding geodesic Lagrangian density takes the form

$$\mathcal{L} = \left(\frac{r}{\alpha}\right)^{\frac{1+\sqrt{5}}{2}} \dot{t}^2 - \dot{r}^2 - \left(\frac{r}{\alpha}\right) b^2\dot{\theta}^2 - \left(\frac{r}{\alpha}\right)^{\frac{1-\sqrt{5}}{2}} \dot{z}^2. \tag{7.3.5}$$

The Noether symmetry generators are

$$\begin{aligned} &\mathbf{Y}_0 = \frac{\partial}{\partial s}, \quad \mathbf{X}_0 = \frac{\partial}{\partial t}, \quad \mathbf{X}_1 = \frac{\partial}{\partial \theta}, \quad \mathbf{X}_2 = \frac{\partial}{\partial z}, \\ &\mathbf{X}_3 = s\frac{\partial}{\partial s} + \frac{t(3-\sqrt{5})}{8}\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{\theta}{4}\frac{\partial}{\partial \theta} + \frac{z(3+\sqrt{5})}{8}\frac{\partial}{\partial z}. \end{aligned} \tag{7.3.6}$$

The conservation laws corresponding to these Noether symmetries are given in Table 7.2.

Table 7.2: First Integrals

Gen	First Integrals
$\mathbf{Y}_0$	$\phi_0 = \mathcal{L}$
$\mathbf{X}_0$	$\phi_1 = -2\dot{t}(\frac{r}{\alpha})^{\frac{1-\sqrt{5}}{2}}$
$\mathbf{X}_1$	$\phi_2 = 2b^2\dot{\theta}\frac{r}{\alpha}$
$\mathbf{X}_2$	$\phi_3 = 2\dot{z}(\frac{r}{\alpha})^{\frac{1+\sqrt{5}}{2}}$
$\mathbf{X}_3$	$\phi_4 = s\mathcal{L} - \frac{t\dot{t}(3-\sqrt{5})}{4}(\frac{r}{\alpha})^{\frac{1-\sqrt{5}}{2}} + r\dot{r} + \frac{b^2 r\theta\dot{\theta}}{2\alpha} + \frac{z\dot{z}(3+\sqrt{5})}{4}(\frac{r}{\alpha})^{\frac{1+\sqrt{5}}{2}}$

The surviving components of the Riemann curvature tensor for spacetime given in equation (7.3.4) are

$$\begin{aligned} &R_{0101} = -\frac{(\frac{r}{\alpha})^{\frac{1+\sqrt{5}}{2}}(1-\sqrt{5})}{8r^2}, \quad R_{0202} = -\frac{(\frac{r}{\alpha})^{\frac{1+\sqrt{5}}{2}}(1+\sqrt{5})}{8\alpha r}, \quad R_{0303} = \frac{1}{4\alpha r}, \\ &R_{1212} = -\frac{1}{4\alpha r}, \quad R_{1313} = \frac{(\frac{r}{\alpha})^{\frac{1+\sqrt{5}}{2}}(1-\sqrt{5})}{8r^2}, \quad R_{2323} = \frac{(\frac{r}{\alpha})^{\frac{1-\sqrt{5}}{2}}(1-\sqrt{5})}{8\alpha r}. \end{aligned}$$

7.4 Spherically Symmetric Spacetime and Noether Symmetries

In Chapter 6 a complete classification of spherically symmetric static spacetimes by Noether symmetries is given. It is seen that the action of spherically symmetric static spacetimes may have $5, 6, 7, 9, 11$, or 17 Noether symmetries. A few examples of spacetimes for which the action of Lagrangian having minimal (i.e. 5) Noether symmetries are given in Table 6.1. Briefly, there appear two classes admitting six, five classes having seven, four classes admitting nine (including the Bertotti-Robinson and the Einstein metrics), whereas only one class of eleven (which is the famous de-Sitter spacetime) and one class of seventeen (Minkowaski spacetime) Noether symmetries. Just like plane symmetric static spacetimes, for spherically symmetric static spacetimes the minimum number of the Noether symmetries are five and the maximum number of the Noether symmetries are seventeen, while the minimum number of isometries is four and the maximum number of isometries is ten.

There are three new cases of spherically symmetric static spacetimes that we have not seen in the literature. These spacetimes are given by equations (6.6.1), (6.6.2) and (6.7.1). The Ricci scalar, and non-vanishing components of the Ricci tensor and the Riemann curvature are given below respectively:

$$(6.6.1): \quad R_s = \frac{2(a^2-1)}{a^2};\ R_{00} = -\frac{\sec^2\frac{r}{a}}{a^2},\ R_{11} = \frac{\sec^2\frac{r}{a}}{a^2},\ R_{22} = -1,\ R_{33} = -\sin^2\theta;$$
$$R_{0101} = -\frac{\sec^4\frac{r}{a}}{a^2},\ R_{2323} = -\sin^2\theta;$$
$$(6.6.2): \quad R_s = \frac{(\alpha^2-1)}{\alpha^2};\ R_{00} = -\frac{1}{r^2}, \quad R_{11} = \frac{1}{r^2},\ R_{22} = -1,\ R_{33} = -\sin^2\theta;$$
$$R_{0101} = -\frac{\alpha^2}{r^4}, \quad R_{2323} = -\sin^2\theta.$$
$$(6.7.1): \quad R_s = 2;\ R_{22} = -1, \quad R_{33} = -\sin^2\theta;\ R_{2323} = -\sin^2\theta.$$

7.5 Spherically Symmetric Vacuum Solutions of EFEs

The general metric for spherically symmetric static spacetime is

$$ds^2 = e^{\nu(r)}dt^2 - e^{\mu(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{7.5.1}$$

Equating the non-vanishing components of Ricci curvature tensor to zero we have

$$\begin{aligned} R_{00} &= \mu'(r)\nu'(r)r - \nu'^2(r)r - 2\nu''(r)r - 4\nu'(r) = 0,\\ R_{11} &= \mu'(r)\nu'(r)r - \nu'^2(r)r - 2\nu''(r)r + 4\mu'(r) = 0,\\ R_{22} &= \mu'(r)r - \nu'(r)r + 2e^{\mu(r)} - 2 = 0,\\ R_{33} &= \sin^2\theta R_{22}. \end{aligned} \tag{7.5.2}$$

The solution of this system is

$$\nu(r) = (1 - \frac{m}{r}), \quad \mu(r) = \frac{1}{(1-\frac{m}{r})}. \tag{7.5.3}$$

These values of $\nu(r)$ and $\mu(r)$ defined the famous Schwarzschild spacetime.

During the classification of spacetimes by Noether symmetries it is observed that symmetries where arc length parameter "s" is not involved, are isometries and all the homothetic vectors appear with an additional scaling term $s\frac{\partial}{\partial s}$. It is also observed here that the only Noether symmetry, other than isometries and homotheties, that does not have any gauge term is $\frac{\partial}{\partial s}$ which corresponds to the Lagrangian density of the metric in each case. Further, in the absence of proper homothety, spacetimes with $m-$dimensional sections of zero curvature admit m Noether symmetries of the form $s\frac{\partial}{\partial x^i}, \quad i = 1, 2, ..., m$, [33].

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